Midterm 2

12:40 - 2:00, April 3, 2014

Instructions:

- There are **four** questions on this midterm. Answer each question part in the space below it. You can use the additional blank pages at the end for rough work if necessary. **Do NOT write answers on the back of any sheet or in the additional blank pages. Any such writing will not be scanned in to Pandagrader, and will NOT be graded.**
- You can use any facts in the lecture notes without deriving them again. **You must prove any facts which were not derived in lecture or homework.**
- None of the questions requires a very long or complicated answer, so avoid writing too much! Unclear or unnecessarily long solutions may be penalized.
- The approximate credit for each question part is shown in the margin (44 points total + 5 points extra credit).
- You may use one double-sided sheet of notes. **No calculators are allowed** (or needed).

Your Name: **Your Lab Section:**

Your Student ID: **Exam Room:**

Name of Student on Your Left:

Name of Student on Your Right:

(Optional & For statistical purposes only)

Percent of lectures you attend:

If less than 75%, why?

For official use – do not write below this line!

Q1	Q2	Q3	Q4	Total

1. (*Warm-up: Easy ones* – 10 points)

a) (1 point) Show the following: if $x(n) \overset{\mathcal{FT}}{\longleftrightarrow} X(\omega)$, then $x(-n) \overset{\mathcal{FT}}{\longleftrightarrow} X(-\omega)$.

Answer:

$$\sum_{n=-\infty}^{\infty} x(-n)e^{-i\omega n} = \sum_{n=-\infty}^{\infty} x(n)e^{i\omega n} = X(-\omega)$$

b) (1 point) A good discrete-time low-pass filter with frequency response $H(\omega)$ has relatively small $|H(\omega)|$ at frequencies near $\omega = 2\pi$. True or false? Explain your answer.

Answer: False. DTFT periodicity implies $X(2\pi) = X(0)$, so frequencies near zero are attenuated.

c) (1 point) What is the main difference between the DTFT and the DFT?

Answer: The DTFT is for aperiodic signals of infinite duration. The DFT is for finite-duration or periodic signals. Alternatively: DTFT is a continuous function, DFT is discrete.

d) (2 points) Suppose $|\gamma| < 1$ and when $\gamma^n u(n)$ is input into a discrete-time LTI system, $y(n)$ comes out. What is the impulse response for this system in terms of $y(n)$?

Answer: It is easy to see it is $y(n) - \gamma y(n-1)$. Indeed, if $x(n) = \gamma^n u(n)$, then $\delta(n) = x(n) - \gamma x(n-1)$, so $h(n) = y(n) - \gamma y(n-1)$ by linearity.

e) (2 points) Consider a causal discrete-time LTI system with impulse response $h(n) \geq 0$ for all n. Suppose we input the signal $x(n) = h(n)$ and get the output

$$y(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 2 \\ 2 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is $h(n)$?

Answer: $h(n) = 1$ for $n = 0, 1$, and zero otherwise. This is easy to see by drawing a picture and verifying by doing the simple convolution.

f) (3 points) As we saw in the homework, an RC-circuit which is described by the differential equation $y(t) = x(t) - RCy'(t)$ has an impulse response

$$h(t) = \frac{1}{RC} e^{-t/(RC)} u(t).$$

What is the output $y(t)$ for the input $x(t) = \cos(2\pi f t)$? Is this a high-pass filter, low-pass filter, or neither? Explain.

Answer: Since $e^{-at}u(t) \overset{\mathcal{FT}}{\longleftrightarrow} \frac{1}{a+i\omega}$, it follows by linearity that the frequency response is $H(\omega) = \frac{1}{1+i\omega RC}$. Thus, we have:

$$|H(\omega)| = \frac{1}{\sqrt{(1 - i\omega RC)(1 + i\omega RC)}} = \frac{1}{\sqrt{1 + (\omega RC)^2}}.$$

Further, the angle is given by

$$H(\omega) = \tan^{-1}(-\omega RC) = -\tan^{-1}(\omega RC).$$

So, the output is

$$\frac{\cos(2\pi f t - \tan^{-1}(2\pi f RC))}{\sqrt{1 + (2\pi f RC)^2}}.$$

2. (*We're getting the band back together!* – 11 points) In the guitar lab, you looked at discrete-time comb filters of the form $y(n) = x(n) + \alpha y(n - N)$. In this problem, we will look at a continuous-time comb filter which is described by the equation:

$$y(t) = x(t) + \frac{9}{10} y(t - \tau).$$

a) (2 points) What is the frequency response $H(\omega)$ for this system?

Answer: Easy to check by inputting $e^{i\omega t}$:

$$H(\omega)e^{i\omega t} = e^{i\omega t} + \frac{9}{10} H(\omega)e^{i\omega(t-\tau)}$$

Cancel out the $e^{i\omega t}$, and rearrange to get

$$H(\omega) = \frac{10}{10 - 9e^{-i\omega\tau}}$$

b) (2 points) Suppose we want to synthesize a "High E" note along with its harmonics by inputting a brief pulse. A "High E" is a tone at 330Hz. How should we select the delay τ to accomplish this?

Answer: We want the bottom to hit its minimum when $\omega = 2\pi \times 330$, so we just put $\tau = 1/330$.

c) (3 points) The infinite number of harmonics would probably be really annoying to listen to. Suppose we change the system to be

$$y(t) = x(t) + \frac{9}{10}\int_{-\infty}^{t} y(s-\tau)e^{\beta(s-t)}ds.$$

for some $\beta > 0$. What is the frequency response for this new system?

Answer: The system $\frac{9}{10}\int_{-\infty}^{t} y(s)e^{\beta(s-t)}ds$ has impulse response $\frac{9}{10}e^{-\beta t}u(t)$, so the frequency response for this part is $F(\omega) = e^{-i\omega\tau}\frac{9/10}{\beta+i\omega}$, where the $e^{-i\omega\tau}$ comes from the time-shift property since we input $y(t-\tau)$ to this system and not $y(t)$. Now, recognize that this is a feedback system with $G(\omega) = 1$, and just plug into the equation to get

$$H(\omega) = \frac{G(\omega)}{1 - G(\omega)F(\omega)} = \frac{1}{1 - e^{-i\omega\tau}\frac{9/10}{\beta+i\omega}} = \frac{\beta + i\omega}{\beta + i\omega - e^{-i\omega\tau}9/10}.$$

d) (2 points) Do the high-frequency harmonics disappear like we want them to? Specifically, what is the amplitude of the k^{th} harmonic (i.e., $k \times 330$Hz) in terms of β?

Answer: The high harmonics don't disappear. Looking at the magnitude, $|H(\omega)|$ approaches 1 as ω grows. To see the gain for the k-th harmonic, just plug in $\omega = k2\pi 330$ to get:

$$H(k2\pi 330) = \frac{\beta + ik2\pi 330}{\beta + ik2\pi 330 - e^{-i\omega\tau}9/10} = \frac{\beta + ik2\pi 330}{\beta + ik2\pi 330 - 9/10},$$

where the second equality follows if we use $\tau = 1/330$ from before. Either would have been an acceptable answer, but you could also find the squared magnitude if you want a real number:

$$|H(k2\pi 330)|^2 = \frac{\beta^2 + (k2\pi 330)^2}{(\beta - 9/10)^2 + (k2\pi 330)^2}$$

e) (2 points) Suggest an input signal $x(t)$ for the system in part (c) which will synthesize a guitar-like output containing only frequencies less than 1000Hz. Your input doesn't necessarily need to be physically realizable.

Answer: In the frequency domain, the output will be $H(\omega)X(\omega)$. So, just put in

$$x(t) = \frac{\tau}{2\pi} \operatorname{sinc}\left(\frac{\tau t}{2}\right),$$

which has a CTFT that is a box with support $\omega \in (-\tau/2, \tau/2)$ (we used this fact several times in lecture, but we could derive it again without too much effort). Select $\tau = 4\pi \times 1000$ to make $Y(2\pi f) = 0$ for frequencies above 1000Hz.

3. (*May the odds be ever in your favor* – 10 points) A signal $x_e(t)$ is *even* if $x_e(t) = x_e(-t)$. A signal $x_o(t)$ is *odd* if $x_o(t) = -x_o(-t)$. A real signal $x(t)$ can always be expressed as the sum of even and odd components

$$x(t) = x_e(t) + x_o(t).$$

a) (3 points) If $x(t) \overset{\mathcal{FT}}{\longleftrightarrow} X(\omega)$, show that for real $x(t)$,

$$x_e(t) \overset{\mathcal{FT}}{\longleftrightarrow} \mathrm{Re}[X(\omega)] \qquad \text{and} \qquad x_o(t) \overset{\mathcal{FT}}{\longleftrightarrow} i \times \mathrm{Im}[X(\omega)].$$

Answer:

$$X_e(\omega) = \int_{-\infty}^{\infty} x_e(t)e^{-i\omega t}dt = \int_0^{\infty} x_e(t)\left(e^{-i\omega t} + e^{-i\omega t}\right)dt = \int_0^{\infty} x_e(t) 2\cos(\omega t)dt,$$

which is real. Similarly,

$$X_o(\omega) = \int_{-\infty}^{\infty} x_o(t)e^{-i\omega t}dt = \int_0^{\infty} x_o(t)\left(e^{-i\omega t} - e^{-i\omega t}\right)dt = i\int_0^{\infty} x_o(t) 2\sin(\omega t)dt,$$

which is imaginary. By linearity, $X(\omega) = X_e(\omega) + X_o(\omega)$, so the claim must be true.

b) (2 points) Find the Fourier transform of $y(t) = e^{-|t|}$.

Answer: $e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$, and we already saw that $e^{-t}u(t) \overset{\mathcal{FT}}{\longleftrightarrow} \frac{1}{1+i\omega}$. Likewise, $x(-t) \overset{\mathcal{FT}}{\longleftrightarrow} X(-\omega)$, so by linearity

$$Y(\omega) = \frac{1}{1+i\omega} + \frac{1}{1-i\omega} = \frac{2}{1+\omega^2}$$

c) (2 points) Find the inverse Fourier transform of

$$Z(\omega) = 2\frac{i\omega}{1+\omega^2}.$$

Answer: Note that $Z(\omega) = i\omega X(\omega)$ from the previous problem, and we know that $x'(t) \overset{\mathcal{FT}}{\longleftrightarrow} i\omega X(\omega)$, so we just differentiate to get:

$$z(t) = x'(t) = -e^{-t}u(t) + e^{t}u(-t) = -e^{-|t|}u(t) + e^{-|t|}u(-t).$$

d) (2 points) Find the even and odd components of the signal $x(t) = e^{-t}u(t)$.

Answer: You could do this brute-force, or you could use the previous parts to realize that $x_e(t) = \frac{1}{2}y(t)$ and $x_o(t) = -\frac{1}{2}z(t)$.

e) (1 point) Let $x_o(t)$ be an arbitrary odd signal. If you input $x_o(t)$ into a causal LTI system, do we expect the output to be even? odd? neither? Or, is there not enough information to answer this question? Justify your response.

Answer: A causal LTI system has impulse response which is neither odd, nor even. So, $H(\omega)X(\omega)$ will have both real and imaginary parts, despite $X(\omega)$ being imaginary. Thus, the output is neither odd nor even.

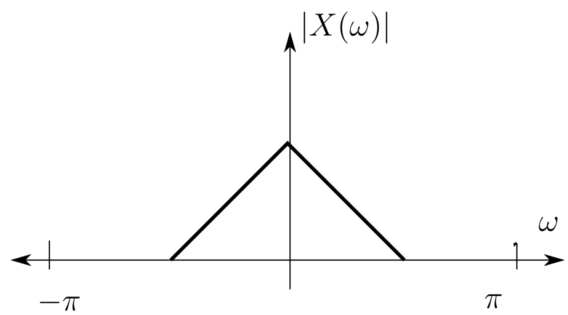
4. (*Here we go again* – 13 points) For a discrete-time signal $x(n)$, the "time-expanded" signal $x_5(n)$ is given by

$$x_5(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-5k).$$

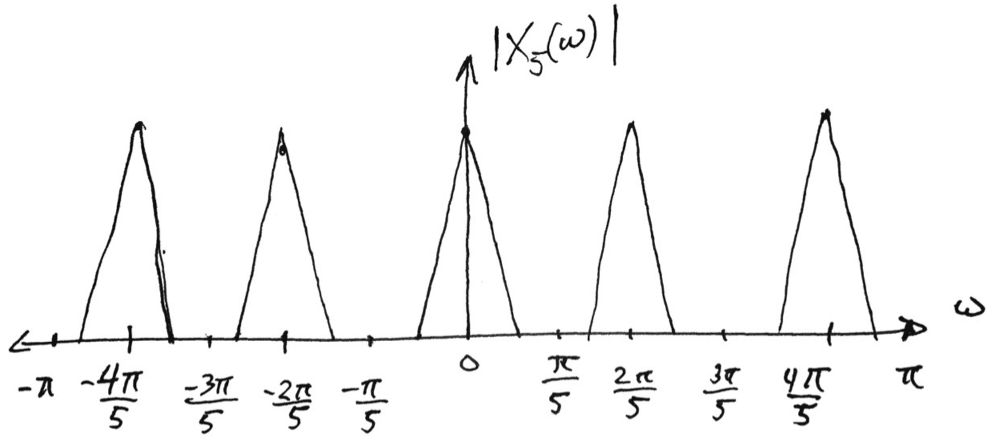
a) (2 points) Is the system taking $x(n)$ to $x_5(n)$ linear? Is it time-invariant?

Answer: It is linear, but not time-invariant.

b) (2 points) For a discrete-time signal $x(n)$ with $x(n) \overset{\mathcal{FT}}{\longleftrightarrow} X(\omega)$, and $|X(\omega)|$ shown below, sketch $|X_5(\omega)|$ (i.e., the magnitude of the DTFT of $x_5(n)$) on the interval $[-\pi, \pi]$.



Answer: Using the scaling property that we saw on the HW, we know the picture should be squished by a factor of 5. Keeping in mind periodicity of the DTFT, the picture should have 5 triangles in it, centered at $2\pi k/5$, where $k = -2, -1, 0, 1, 2$.



c) (1 point) For the signal in part (b), is $X_5(\omega)$ periodic? If so, what is the period.

Answer: Yes. It has period $2\pi/5$.

d) (2 points) Consider the discrete-time impulse response

$$h(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

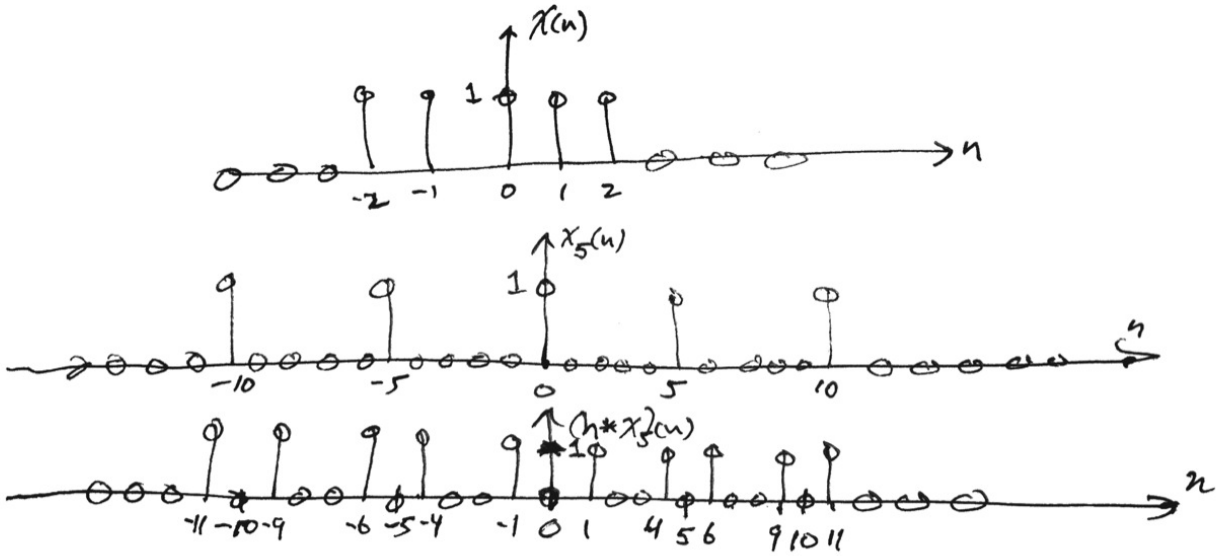
What is $H(\omega)$?

Answer: $H(\omega) = e^{-i\omega} + e^{i\omega} = 2\cos(\omega)$.

e) (3 points) For the signal

$$x(n) = \begin{cases} 1 & \text{if } |n| \le 2 \\ 0 & \text{otherwise,} \end{cases}$$

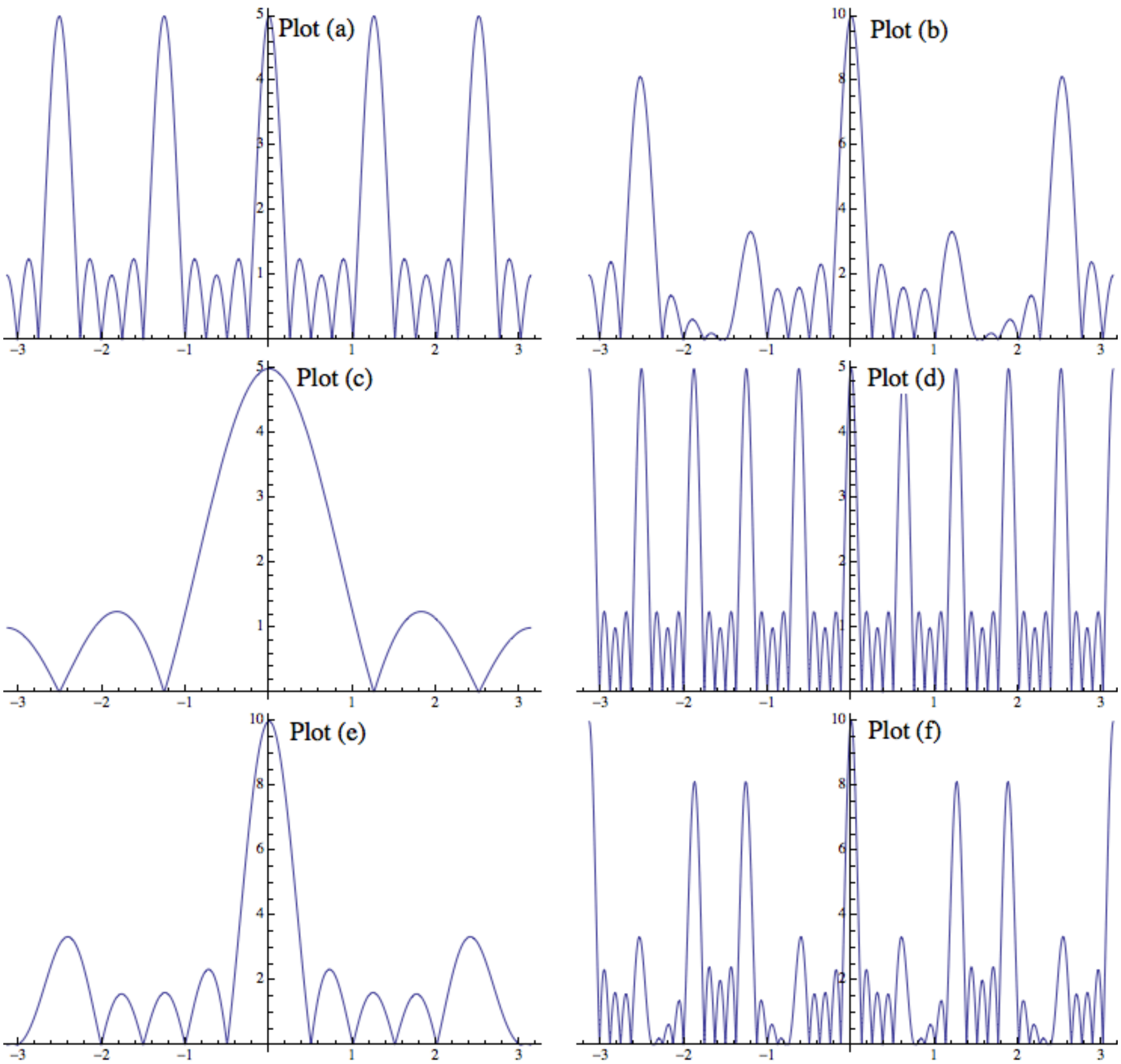
sketch $x(n)$, $x_5(n)$, and $(x_5 * h)(n)$ (on three separate, clearly labeled plots).



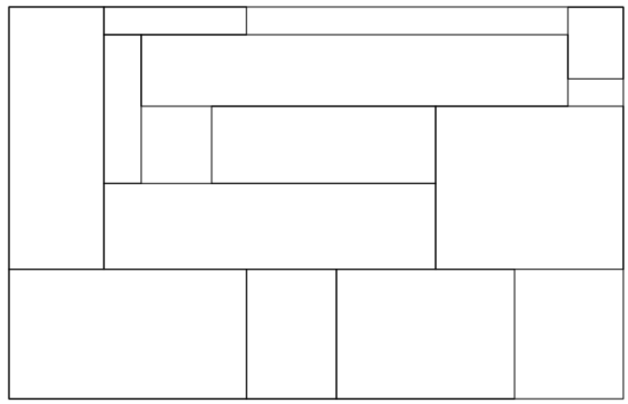
f) (3 points) Using $x(n)$ from part (e), identify the DTFT's of $x(n)$, $x_5(n)$ and $(x_5 * h)(n)$ from among the given plots. Write "not present" if the specified function is not shown.

- The magnitude of the DTFT of $x(n)$ is given by plot ________________.
- The magnitude of the DTFT of $x_5(n)$ is given by plot ________________.
- The magnitude of the DTFT of $(x_5 * h)(n)$ is given by plot ________________.

Answer: (c), (a), (b). Basically, we know $X_5(\omega)$ is $X(\omega)$ squished by a factor of 5, so that gives (c) and (a) (so, you could do it even if you forgot that the FT of a box is a sinc). $H(\omega)X_5(\omega) = 2X_5(\omega)\cos(\omega)$, and there is only one plot that looks like it is plot (a) with the magnitudes multiplied by $2|\cos(\omega)|$. So, final answer is (b).



5. (**Extra Credit:** *Mind = Blown* – 5 points) Suppose a large rectangle is covered by several smaller rectangles, as shown in the example below.



If each of the smaller rectangles has at least one dimension (either a length or a width) which is an integer number of centimeters, show that the large rectangle must also have a dimension which is an integer number of centimeters.

Hint: consider the 2-D Fourier transform

$$F(\omega, \xi) = \iint f(x, y) e^{-i2\pi(x\omega + y\xi)} \mathrm{d}x\mathrm{d}y.$$

Answer: Per the hint, consider taking the transform over a function $f(x, y)$ which is equal to one when $a \le x \le b$ and $c \le y \le d$, and zero otherwise. Then, we have for $\omega = \xi = 1$:

$$F(1,1) = \int_a^b e^{-i2\pi x} \mathrm{d}x \int_c^d e^{-i2\pi y} \mathrm{d}y \tag{1}$$

$$= \Big(-i \left(e^{-i2\pi a} - e^{-i2\pi b} \right) \Big) \Big(-i \left(e^{-i2\pi c} - e^{-i2\pi d} \right) \Big) \tag{2}$$

$$= e^{-i2\pi(a+c)} \left(1 - e^{i2\pi(a-b)} \right) \left(1 - e^{i2\pi(c-d)} \right). \tag{3}$$

Now, we see that $\underline{F(1,1) = 0 \text{ if and only if } (b-a) \text{ or } (d-c) \text{ is an integer}}$. By linearity, the Fourier transform taken over the big rectangle is equal to the sum of the Fourier transforms taken over the small rectangles. The Fourier transforms taken over the small rectangles are each zero (since each has an integer-length dimension by assumption), implying the Fourier transform taken over the big rectangle must also be zero. Hence, one of its sides must be an integer in length!

End of Exam