Midterm 2 Solutions

a) The asymmetry in the twin paradox comes from the fact that the twin travelling to the star must accelerate

also accepted: The interval (therefore proper time) is different for the paths

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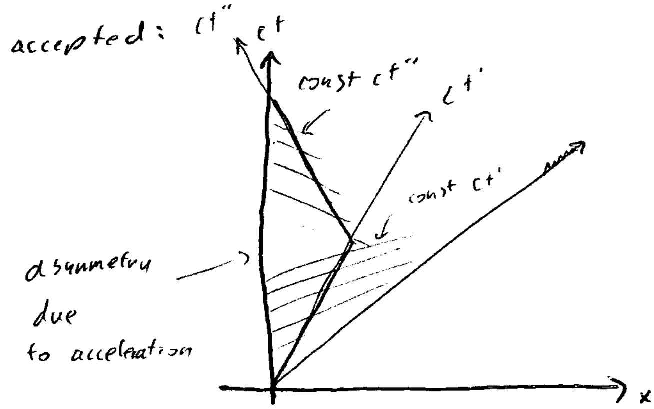
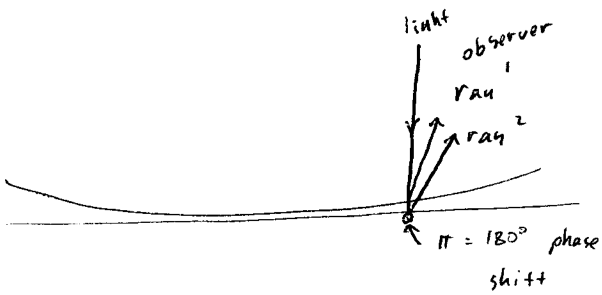


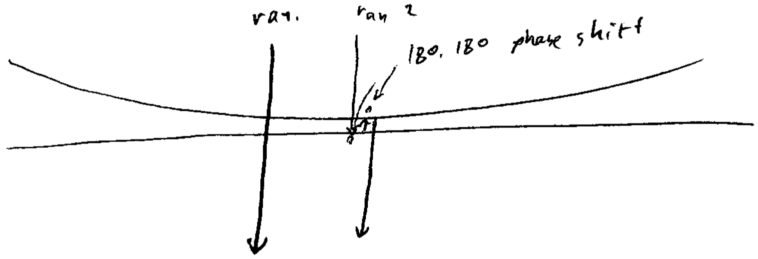
+5 any of these answers

b) In reflection: the lower path glass-air-glass (ray 2) picks up a 180° phase shift at the reflection - out of phase



→ DARK

In transmission: the 2nd ray reflects twice - back in phase with incident ray.



→ BRIGHT

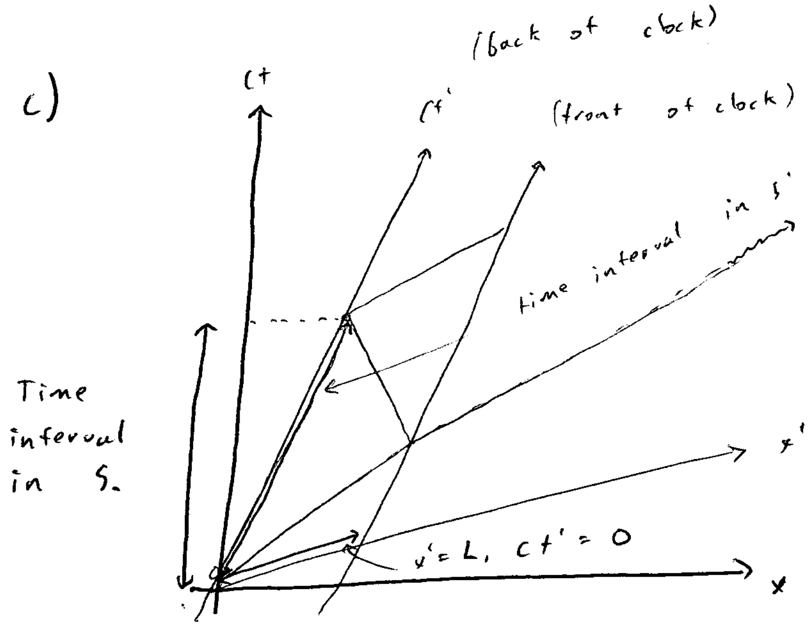
+2 reflection dark

+1 transmission bright

+2 explanation

(1 for explaining each)

c)



+5 correct diagram

d) acceptable reasons:

→ spherical abberation (distortion of the image)

→ diffraction limit for small lenses

$$\Delta\theta = 1.22\,\frac{\lambda}{D}$$

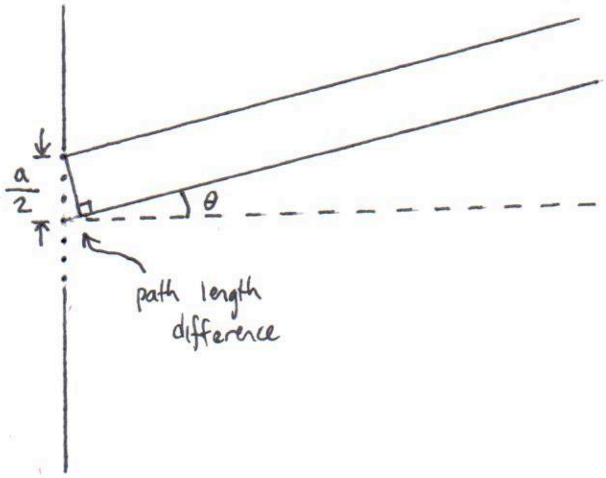
→ for small enough focal lengths the thin lens approximation breaks

+3 one answer

+2 second answer

2) As specified in the problem statement, solutions required complete arguments utilizing path length differences (or phasors) for full credit.

a) Consider multiple wavefronts passing through single slit:



Assuming $D \gg a$, paths to screen are nearly parallel

Diffraction minima will occur when wavefront at edge of slit destructively interferes with wavefront at center of slit. (And likewise every wavefront will have a canceling "partner" halfway down slit.)

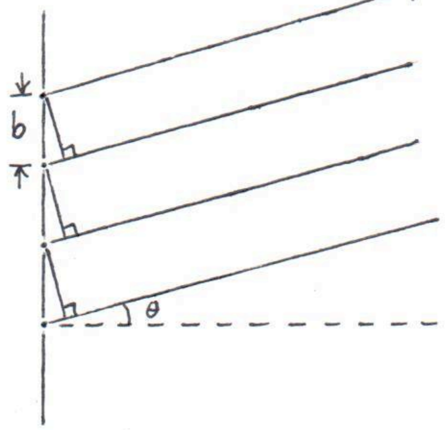
$$\Rightarrow \text{path length difference} = \frac{a}{2}\sin\theta = (m+\tfrac{1}{2})\lambda \quad ; \quad m = 0,1,2\ldots$$

• first minimum is $m=0$

$$\Rightarrow \frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

$$\Rightarrow \theta \approx \sin\theta = \frac{\lambda}{a} = \frac{600\times10^{-9}\,m}{2\times10^{-6}\,m} = \boxed{0.3 \text{ rad} \approx 17^\circ}$$

b) Now consider wavefronts passing through each slit:



Principal maxima (brightest fringes) will occur when adjacent slits constructively interfere.
(differ by whole wavelengths)

path length difference between adjacent slits $= b\sin\theta$

$$\Rightarrow b\sin\theta = n\lambda \;;\; n = 0, 1, 2, \ldots$$

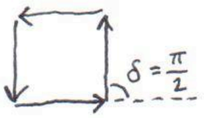
But $\sin\theta = \dfrac{n\lambda}{b} = \dfrac{\lambda}{a}$ when $n = 3$ (because $b = 3a$),

therefore only $n = 0, 1, 2$ are within central diffraction band.

$$\left.\begin{array}{ll} n=0: & \boxed{\theta = 0} \\ n=1: & \theta \sim \dfrac{\lambda}{b} = \dfrac{600\text{ nm}}{6\ \mu\text{m}} = \boxed{0.1\text{ rad} \approx 5.7^\circ} \\ n=2: & \theta \sim \dfrac{2\lambda}{b} = \dfrac{2(600\text{ nm})}{6\ \mu\text{m}} = \boxed{0.2\text{ rad} \approx 11.5^\circ} \end{array}\right\} \text{5 total bright fringes}$$

* Note: there will also be secondary maxima corresponding to constructive interference between non-adjacent slits. It was not necessary to account for these for full credit.

c) The first interference minimum will occur when the phase difference between adjacent slits is $\delta = \dfrac{2\pi}{N} = \dfrac{\pi}{2}$.



Since $\delta = \dfrac{2\pi}{\lambda} \times$ path length difference,

this corresponds to a path length diff. between adjacent slits of: $b\sin\theta = \dfrac{\lambda}{2\pi} \cdot \dfrac{\pi}{2} = \dfrac{\lambda}{4}$.

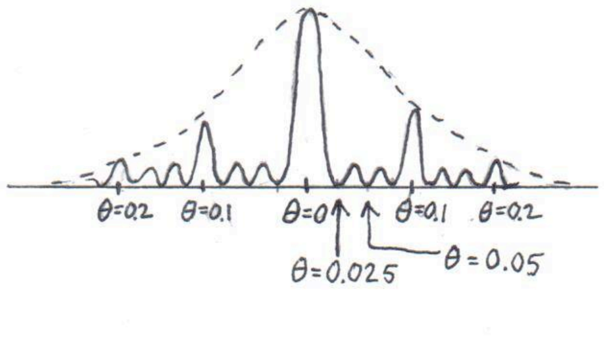
$$\Rightarrow \theta \sim \sin\theta = \frac{\lambda}{4b} = \frac{600\text{ nm}}{4(6\ \mu\text{m})} = \boxed{0.025\text{ rad} \approx 1.4^\circ}$$

* Note: if you just applied the destructive interference condition to the adjacent slits, you got:

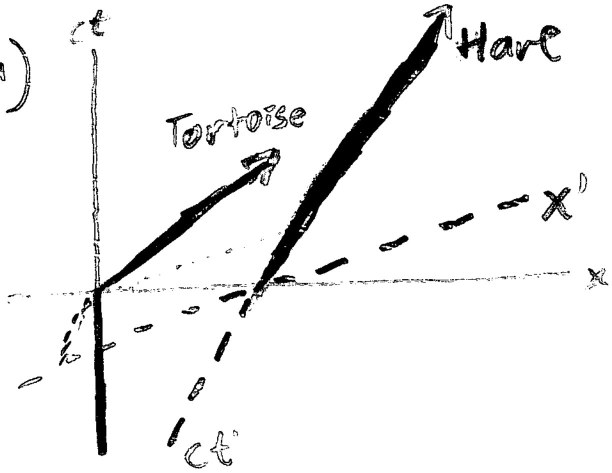
$$b\sin\theta = \frac{\lambda}{2} \Rightarrow \sin\theta = \frac{\lambda}{2b}$$

which IS a minimum, but not the first one.

The interference pattern looks like this:



3a)



The hare's worldline is equivalent to her ct' axis (as it corresponds to $x'=0$); her x' axis is skew from that. Because the x' axis has positive slope in the unprimed coordinates, and its origin is shifted to the right, then the tortoise's start is above the x' axis, i.e. $ct' > 0$. Thus the tortoise had a slow start in the hare's reference frame

b) In the Earth frame, the hare travels a distance of $L/2$ at speed $c/3$, thus the race took a time of $(L/2)/(c/3) = \frac{3L}{2c}$.

In the Earth frame, the tortoise finished the race in the same amount of time, $\frac{3L}{2c}$, but had to cover a distance of L. Thus he moved at a speed of $L/\left(\frac{3L}{2c}\right) = \frac{2}{3}c$.

Therefore, relative to the hare:

$$u' = \frac{u-v}{1-\frac{uv}{c^2}} = \frac{\frac{2}{3}c - \frac{1}{3}c}{1-\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)} = \frac{3}{7}c$$

(The minus sign is used because the tortoise and the hare are moving in the same direction in the Earth frame)

c) The tortoise can run no faster than c. Therefore, it will always take a time of at least $\frac{L}{c}$ for the tortoise to finish the race, and if the hare can finish in less than that time, she's guaranteed to win. Thus, her speed needs to be at least $(L/2)/(L/c) = \frac{c}{2}$.

To find the distance to the finish line in the hare's frame, it suffices to use length contraction, since she is observing two events at different locations but at the same time in her reference frame. Hence:

$$D' = \frac{D}{\gamma} = \frac{L/2}{1/\sqrt{1-(v/c)^2}} = \sqrt{1-(\tfrac{1}{2})^2} \cdot \tfrac{1}{2} L = \frac{\sqrt{3}}{4} L$$

d) Here, the time dilation formula is applicable, since what's being observed is equivalent to two clocks which are each stationary in some reference frame.

$$T = \gamma T'$$

However, γ is still unknown! Fortunately, it can be solved for via the speed:

$$v = \frac{L}{T} \Rightarrow \gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} = \frac{1}{\sqrt{1-(\frac{L}{cT})^2}}$$

Thus:

$$T' = T\sqrt{1-(\tfrac{L}{cT})^2} = \sqrt{T^2-(\tfrac{L}{c})^2}$$

$$T'^2 = T^2 - (\tfrac{L}{c})^2$$

$$T^2 = T'^2 + (\tfrac{L}{c})^2$$

$$T = \sqrt{T'^2 + (\tfrac{L}{c})^2}$$

Alternatively: $(c\Delta t)^2 - (\Delta x)^2$ is an invariant quantity.

In the primed frame, $\Delta t = T'$ and $\Delta x = 0$ (as the tortoise is not moving in his own frame), while in the unprimed frame, $\Delta x = L$. Thus:

$$(cT')^2 - 0^2 = (cT)^2 - L^2$$

$$T = \frac{1}{c}\sqrt{(cT')^2 + L^2}$$

Problem 3 Rubric

a) · 0 points for citing time dilation/length contraction only

· 7 points for writing and using the Lorentz transform, 3 more points for obtaining and interpreting the correct result

OR

· 7 points for drawing a correct spacetime diagram with explanation, 3 more points for a correct result

b) 3 points for obtaining the tortoise's speed in the Earth frame, 7 more points for a correct velocity transformation

c) 8 points for finding the correct speed and using length contraction, 2 more points for arguing why length contraction is applicable

d) 3 points for writing time dilation, 1 more point for arguing why time dilation is applicable, 6 more points for relating T, L, and v, and eliminating v and γ to obtain T

4a) Necessary energy is $m_{total}c^2$ to create particles at rest.

$$\boxed{E = 2m_0c^2}$$

b) Energy and momentum are both conserved in a collision. However,

$$E_{photon} = pc$$

$$E_{e} = \sqrt{p^2c^2 + m^2c^4} > pc$$

Thus a single photon cannot create an electron and positron by itself.

c) Because all products move together and have the same mass, each must carry 1/3 the momentum of the initial photon. Then

$$pc + m_0c^2 = E_{initial} = E_{final} = 3\sqrt{m_0^2c^4 + \left(\frac{pc}{3}\right)^2}$$

$$p^2c^2 + 2m_0pc^2 + m_0^2c^4 = 9m_0^2c^4 + p^2c^2$$

$$8m_0^2c^4 = 2m_0pc^2$$

$$p = 4m_0c$$

$$\boxed{E_{photon} = 4m_0c^2}$$

4 a)

+4 – recognize $E = mc^2$ for particle at rest

+1 – two particles of mass m_0

5

2 pts for energy without minimizing

b)

+3 – show use of energy conservation

+3 – show use of momentum conservation

+4 – greater energy for massive particles at same total momentum

10

c)

+2 – apply conservation of total momentum

+2 – $p_{electron} = \frac{1}{3} p_{photon}$

+2 – apply conservation of total energy

+2 – $E_{photon} = pc$

+2 – $E_{rest} = mc^2$

+2 – $E_{moving} = \sqrt{m^2c^4 + p^2c^2}$ (or γmc^2)

+2 – 3 particles in final state

+1 – carry through algebra to correct answer

15