#1

The field from the sphere can be found using Gauss' Law

$$\int E \cdot dA = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \rho \left(\tfrac{4}{3}\pi R^3\right)$$

$$\int \vec{E} \cdot d\vec{A} = E(r) \int r^2 \sin\theta \, d\theta \, d\phi$$

$$= E(r) \, 4\pi r^2$$

$$\vec{E}(\vec{r}) = \frac{\rho \left(\frac{4}{3}\pi R^3\right)}{\epsilon_0 \, 4\pi r^2} \hat{r}$$

$$\vec{E}(\vec{r}) = \frac{\rho R^3}{3\epsilon_0 r^2} \hat{r}$$

$$\vec{F} = q\vec{E}$$

$$= \int \lambda \vec{E}(\vec{r}) \, dr$$

$$= \int_R^{R+d} \frac{\lambda \rho R^3}{3\epsilon_0 r^2} dr \, \hat{r} = \frac{\lambda \rho R^3}{3\epsilon_0} \int_R^{R+d} \frac{dr}{r^2} \hat{r}$$

$$= \frac{\lambda \rho R^3}{3\epsilon_0} \left(-\frac{1}{r}\right)\Bigg|_R^{R+d} = \frac{\lambda \rho R^3}{3\epsilon_0} \left(\frac{1}{R} - \frac{1}{R+d}\right)$$

$$\vec{F} = \frac{\lambda \rho}{3\epsilon_0} \left(\frac{R^2 d}{R+d}\right) \hat{r}$$

#2 For a ring of charge with an arbitrary radius r

$$V(x) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+r^2}}$$

our disk is a summation of rings w/ thickness dr

$$dV(x) = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{x^2+r^2}}$$

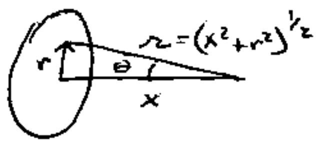
$$Q = \int dq(\theta,r)$$

$$= \int \sigma(\theta,r)\, dA$$

$$= \int \sigma_0 \frac{R}{\cancel{r}} \cancel{r}\, dr\, d\theta$$

$$Q = 2\pi \int \sigma_0 R\, dr$$

$$dq = 2\pi\sigma_0 R\, dr$$



$$\tan\theta = \frac{r}{x}$$

$$\cos\theta = \frac{x}{(x^2+r^2)^{1/2}}$$

$$V(x) = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{dq}{\sqrt{x^2+r^2}}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^R \frac{2\pi\sigma_0 R\, dr}{\sqrt{x^2+r^2}}$$

$$= \frac{\sigma_0 R}{2\epsilon_0} \int_0^R \frac{dr}{\sqrt{x^2+r^2}}$$

let $r = x\tan\theta$

$dr = x\sec^2\theta\, d\theta$

$$= \frac{\sigma_0 R}{2\epsilon_0} \int \frac{x\sec^2\theta\, d\theta}{\sqrt{x^2+x^2\tan^2\theta}} = \frac{\sigma_0 R}{2\epsilon_0} \int \frac{x\sec^2\theta}{x\sqrt{1+\tan^2\theta}}\, d\theta$$

$$= \frac{\sigma_0 R}{2\epsilon_0} \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \frac{\sigma_0 R}{2\epsilon_0} \int \sec\theta \left(\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\right) d\theta$$

$$= \frac{\sigma_0 R}{2\epsilon_0} \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta$$

let $u = \sec\theta + \tan\theta$

$du = (\sec\theta\tan\theta + \sec^2\theta)\, d\theta$

$$= \frac{\sigma_0 R}{2\epsilon_0} \int \frac{du}{u} = \frac{\sigma_0 R}{2\epsilon_0} \ln u \Big|_a^b$$

$$= \frac{\sigma_0 R}{2\epsilon_0} \ln(\sec\theta + \tan\theta)\Big|_a^b$$

$$= \frac{\sigma_0 R}{2\epsilon_0} \ln\left(\frac{(x^2+r^2)^{1/2}}{x} + \frac{r}{x}\right)\Big|_0^R$$

$$V(x) = \frac{\sigma_0 R}{2\epsilon_0} \ln\left[\left(1+\frac{R^2}{x^2}\right)^{1/2} + R/x\right]$$

$$W = Q\,\Delta V$$

$$= Q\left[V(x) - \cancelto{0}{V(\infty)}\right]$$

$$W = \frac{Q\sigma_0 R}{2\epsilon_0} \ln\left[\left(1+\frac{R^2}{x^2}\right)^{1/2} + \frac{R}{x}\right]$$

#3 Since the spheres are far apart we assume that the electric field from one does not effect the charge distribution on the other. The wire guarantees that the two spheres are at the same electric potential

$r_1 = 5\,cm$

$r_2 = 12\,cm$

$E_2 = 2 \times 10^5\ V/m$ at the surface of the large sphere.

$$E_2(r_2) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_2^2} = 2 \times 10^5 \frac{V}{m}$$

$$\frac{Q}{4\pi r_2^2} = \epsilon_0 \cdot 2 \times 10^5 \frac{V}{m}$$

$$\boxed{\sigma_2 = 1.8 \times 10^{-6} \frac{C}{m^2}}$$

$$V(r_2) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\sigma_2 \cdot (4\pi r_2^2)}{r_2}$$

$$= \frac{1}{\epsilon_0} \sigma_2 r_2$$

$$V(r_2) = 2.4 \times 10^4\ V$$

$$V(r_2) = V(r_1)$$

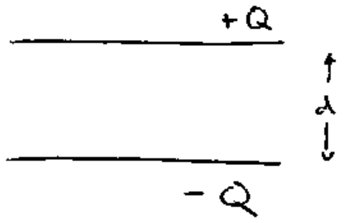
$$V(r_1) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_1} = \frac{1}{4\pi\epsilon_0} \frac{\sigma_1 (4\pi r_1^2)}{r_1} = \frac{\sigma_1 r_1}{\epsilon_0}$$

$$\sigma_1 = \frac{V(r_1)\epsilon_0}{r_1}$$

$$\boxed{\sigma_1 = 4.3 \times 10^{-6} \frac{C}{m^2}}$$

#4

Initial



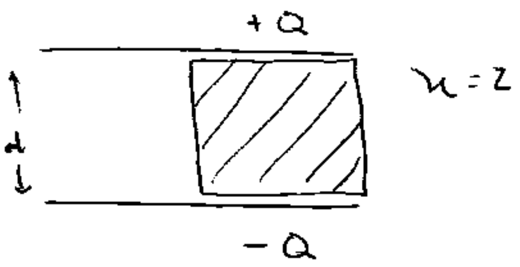
$$\Delta V = V_0$$

$$C_i = \frac{\epsilon_0 A}{d}$$

$$Q = C_0 \Delta V$$

$$Q = \epsilon_0 \frac{A V_0}{d}$$

Final



$$C_{eq} = C_1 + C_2$$

$$= \frac{\epsilon_0 A/2}{d} + \frac{\kappa \epsilon_0 A/2}{d}$$

$$= \frac{\epsilon_0 A}{2d}(1+\kappa) = \frac{3}{2}\frac{\epsilon_0 A}{d}$$

$$= 3/2\, C_0$$

$$Q = C \Delta V$$

$$Q_i = Q_f$$

$$C_0 V_0 = C_{eq} V_f$$

$$C_0 V_0 = 3/2\, C_0 V_f$$

$$\boxed{V_f = \frac{2}{3} V_0}$$

#5

$$P_L = I^2 R_L$$

$$V = I R_{in} + I R_L$$

$$I = \frac{V}{R_{in}+R_L}$$

$$\Rightarrow \quad P = \left(\frac{V}{R_{in}+R_L}\right)^2 R_L$$

To find R_L that maximizes the power, we must evaluate

$$\frac{dP}{dR_L} = 0$$

$$\frac{d}{dR_L}\left(\frac{V}{R_{in}+R_L}\right)^2 R_L = V^2\left[\frac{1}{(R_{in}+R_L)^2} - \frac{2R_L}{(R_{in}+R_L)^3}\right] = 0$$

$$\frac{1}{(R_{in}+R_L)^2} = \frac{2R_L}{(R_{in}+R_L)^3}$$

$$R_{in} + R_L = 2R_L$$

$$\boxed{R_L = R_{in}}$$