Midterm 2 solutions

April 7, 2014

1. (a) For a lenght ℓ section of the cylindrical wire, we calculate the drop in potential:

$$\Delta V = IR = I\frac{\rho\ell}{A}$$

Since the drop in potential is proportional to ℓ, the field is uniform in the longitudinal direction, and equal to

$$E = \frac{\Delta V}{\ell} = \frac{I\rho}{A} = 0.15 \text{ V/m}$$

(b) Considering again the same section of the cylinder, let Q be the charge of the free electrons in this piece of wire, and Δt be the time it takes for all this charge to drift out of it. If n_e is the number density of free electrons, the charge is equal to $n_e e * (\text{volume})$. Then the current can be expressed as:

$$I = \frac{Q}{\Delta t} = \frac{n_e e A\ell}{\Delta t} = n_e e A v_d$$

The number density of electrons is equal to the number density of Ag atoms, since each atom contributes one free electron, so $n_e = d/m_{Ag}$. This gives us a formula for v_d

$$v_d = \frac{m_{Ag} I}{dA\ell} = 5 \times 10^{-4} \text{ m/s}$$

(c) At room temperature, the free electrons are very lightly bound to the ions, so we can assume we have an ideal gas of electrons, and then the average kinetic energy of each electron is given by the equipartition theorem:

$$\left\langle \frac{m_e v^2}{2} \right\rangle = \frac{3}{2} k_B T$$

So the rms speed is

$$v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m_e}} = 10^5 \text{ m/s}$$

(d) $v_{\text{rms}}/v_d = 2 \times 10^8$, so the rms speed of the electrons is much larger than the drift speed. This shows that the conduction of electric current is not accomplished by a significant acceleration of the electrons, but instead by a very small difference in the directional distribution of the electron velocities, between forward and backward directions.

2. Let's calculate the electric field by summing up the contributions due to the charge elements dq on the ring. By symmetry, the components of $\vec{E}$ in the radial directions cancel out, and the only relevant contribution is in the x direction. So at some point at a distance x along the axis we have:

$$|\vec{E}| = E_x = \int dE_x = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{x^2 + r^2} \cos\theta = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{x^2 + r^2} \frac{x}{\sqrt{x^2 + r^2}}$$

where θ is the angle to the charge element dq as seen from the axis. A charge element on the surface of the ring, is given by $dq = \sigma(r) \times d(\text{area}) = \sigma(r) \times r dr d\theta = \beta dr d\theta$. Substituting in the integral

$$|\vec{E}| = \int_{R_1}^{R_2} \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{\beta x dr d\theta}{(x^2 + r^2)^{3/2}} = \frac{\beta}{2\epsilon_0} \int_{R_1}^{R_2} \frac{x dr}{(x^2 + r^2)^{3/2}} = \frac{\beta x}{2\epsilon_0} \left(\frac{R_2}{\sqrt{x^2 + R_2^2}} - \frac{R_1}{\sqrt{x^2 + R_1^2}} \right)$$

(*Alternative solution*)

Let's calculate first the electric potential and then derive the field from that. A charge element on the surface of the ring, given in terms of cylindrical coordinates, is given by $dq = \sigma(r) \times r dr d\theta = \beta dr d\theta$. The potential at a point situated at a distance x along the axis is:

$$V = \int dV = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{r^2 + x^2}} = \frac{\beta}{4\pi\epsilon_0} \int \frac{dr d\theta}{\sqrt{r^2 + x^2}}$$

Integrating over θ gives us

$$V = \frac{\beta}{2\epsilon_0} \int_{R_1}^{R_2} \frac{dr}{\sqrt{r^2 + x^2}}$$

Substituting $u = r/x$ gives us

$$V = \frac{\beta}{2\epsilon_0} \int_{R_1/x}^{R_2/x} \frac{du}{\sqrt{u^2 + 1}} = \frac{\beta}{2\epsilon_0} \left[\ln(R_2 + \sqrt{R_2^2 + x^2}) - \ln(R_1 + \sqrt{R_1^2 + x^2}) \right]$$

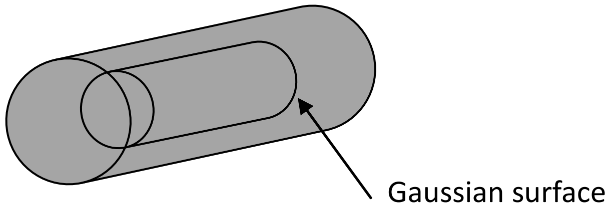
Now that we have the potential we get for the field (which points along the x axis by symmetry)

$$|\vec{E}| = -\frac{dV}{dx} = \frac{\beta x}{2\epsilon_0} \left[\frac{1}{R_1 \sqrt{x^2 + R_1^2} + x^2 + R_1^2} - \frac{1}{R_2 \sqrt{x^2 + R_2^2} + x^2 + R_2^2} \right]$$

which after simplifying gives the answer obtained by the other method.

3.

Because the shape of the object has high symmetry, to calculate the electric field in and out of the cylinder, create Gaussian surface as below:



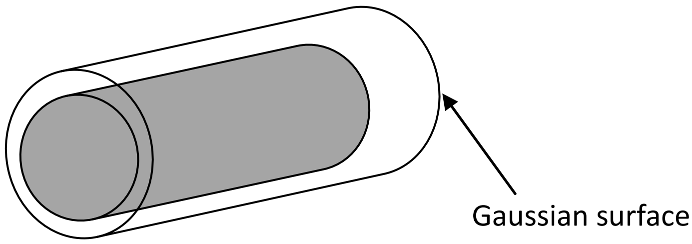
Use Gauss's law to find out E field:

In the cylinder:

$$(2\pi rL)E = \frac{Q_{enc}}{\epsilon_o} = \frac{1}{\epsilon_o}\int_0^r (\rho 2\pi r' L)dr' = \frac{1}{\epsilon_o}\int_0^r (2\pi L\alpha)e^{-\frac{r'}{a}}dr' = \frac{2\pi L\alpha a}{\epsilon_o}(1 - e^{-\frac{r}{a}})$$

$$\Rightarrow \vec{E} = \frac{\alpha a}{\epsilon_o r}(1 - e^{-\frac{r}{a}})\hat{r}$$

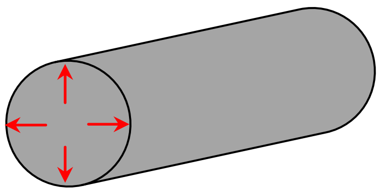
Out of the cylinder:



$$(2\pi rL)E = \frac{Q_{enc}}{\epsilon_o} = \frac{1}{\epsilon_o}\int_0^R (\rho 2\pi r' L)dr' = \frac{1}{\epsilon_o}\int_0^R (2\pi L\alpha)e^{-\frac{r'}{a}}dr' = \frac{2\pi L\alpha a}{\epsilon_o}(1 - e^{-\frac{R}{a}})$$

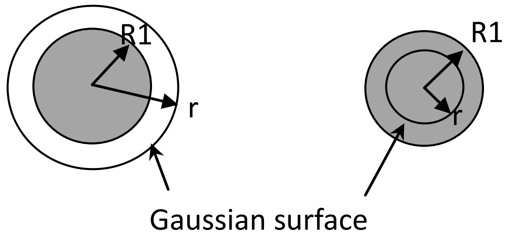
$$\vec{E} = \frac{\alpha a}{\epsilon_o r}(1 - e^{-\frac{R}{a}})\hat{r}$$

Electric field vector:



4.

(a)Use Gauss's law to find out the electric field inside and outside the metallic ball:



When r>R1,

$$(4\pi r^2)E = Q/\epsilon_o$$
$$\Rightarrow \vec{E} = \frac{Q}{\epsilon_o 4\pi r^2}\,\hat{r}$$

when r<R1, all the charges are on the surface of metallic ball, so the electric field inside is zero.

Now calculate the electric potential:

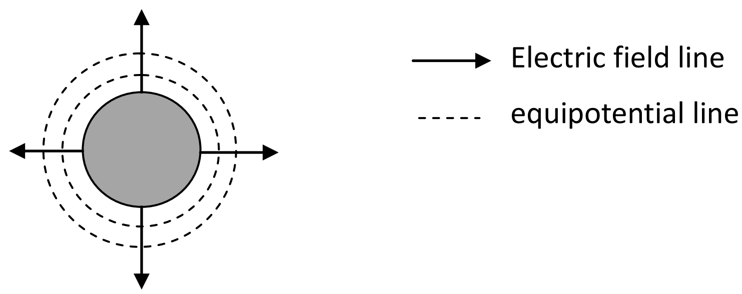
For r>R1:

$$V(r) - V(\infty) = \int_r^{\infty} \frac{Q}{\epsilon_o 4\pi r'^2}\, dr' = \frac{Q}{\epsilon_o 4\pi r}$$
$$V(r) = \frac{Q}{\epsilon_o 4\pi r}$$

For r<R1:

$$V(r) - V(\infty) = \int_r^{R1} 0\, dr' + \int_{R1}^{\infty} \frac{Q}{\epsilon_o 4\pi r'^2}\, dr' = \frac{Q}{\epsilon_o 4\pi R1}$$
$$V(r) = \frac{Q}{\epsilon_o 4\pi R1}$$

(b)



(c)

When two metallic balls are connected with a conducting wire, they share the same electric potential. Suppose there are Q1 charges on ball 1, Q2 charges on ball 2. And Q1+Q2=Q.

Since two balls are far apart from each other, we can take them both as point charges. Therefore the electric potential on ball 1 and ball 2, V1 and V2 respectively, is:

$$V1 = \frac{Q1}{4\pi\epsilon_o R1} + \frac{Q2}{4\pi\epsilon_o d}$$

$$V2 = \frac{Q2}{4\pi\epsilon_o R2} + \frac{Q1}{4\pi\epsilon_o d}$$

Where d is the distance between two balls, b>>R1 and R2.

V1=V2 gives,

$$\frac{Q1}{R1} + \frac{Q2}{d} = \frac{Q2}{R2} + \frac{Q1}{d}$$

Since b>>R1 and R2, Q2/d term and Q1/d term can be neglected.

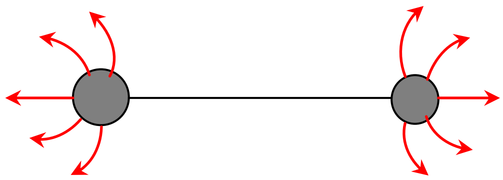
$$\frac{Q1}{R1} = \frac{Q2}{R2}$$

With Q1+Q2=Q, we can find out

$$Q1 = \frac{R1}{R1 + R2} Q$$

$$Q1 = \frac{R2}{R1 + R2} Q$$

(d)



Problem 5

a)

We can find this using Gauss's law. Take a Gaussian surface that is a pillbox around the plate with positive charge. The Gaussian surface should be near the center of the plate, and have a length $l << L$ and some height h. The charge inside will be:

$$Q_{in} = \int \sigma dA = \sigma l^2 = Q\frac{l^2}{L^2}$$

Here, I used the fact that the charge is distributed uniformly on the plate.
Now, for a large sheet of charge, we expect that the electric field should point perpendicularly to it. The best way to see this is to think about the motion of a test charge. A positive test charge will be pushed away from the plate in the direction perpendicular to the plate. However the plate will not cause any motion in a direction parallel to the plate, because to the charge, the plate looks the same in every parallel direction since the plate is large. This means that all the forces coming from charges at various parts of the plate will be canceled by the charges on the other side. Since the field is perpendicular to the plate, the sides of the Gaussian surface perpendicular to the plate have no flux through them, and only the top and bottom surface have flux. Thus:

$$\oint \vec{E}_+ \cdot d\vec{A} = 2\int \vec{E}_+ \cdot d\vec{A} = 2|\vec{E}_+|l^2$$

So, by Gauss's law, the electric field due to one plate in the region between the plates is:

$$\vec{E}_+ = \frac{Q}{2\epsilon_0 L^2}\hat{z}$$

Where $\hat{z}$ is the direction perpendicular to the plane of the sheet of charge.
The negative charge is exactly the same, except the direction is skewed:

$$\vec{E}_- = \frac{Q}{2\epsilon_0 L^2}(\cos(\theta)\hat{z} + \sin(\theta)\hat{x})$$

Note that this looks right as when $\theta \to 0$, the field points in the $\hat{z}$ direction. By superposition principle, the total electric field is:

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{Q}{2\epsilon_0 L^2}((1+\cos(\theta))\hat{z} + \sin(\theta)\hat{x})$$

b)

Take a path starting at the center of one plate and going to the other, along $\hat{z}$. Then:

$$\Delta V = -\int_-^+ \vec{E}\cdot d\vec{l} = \int_0^{d+L\tan(\theta)/2} \frac{Q}{2\epsilon_0 L^2}((1+\cos(\theta))dl = \frac{Q}{2\epsilon_0 L^2}((1+\cos(\theta))(d+L\tan(\theta)/2)$$

So the capacitance is:

$$C = \frac{Q}{\Delta V} = \frac{2\epsilon_0 L^2}{((1+\cos(\theta))(d+L\tan(\theta)/2)} \approx \frac{\epsilon_0 L^2}{d(1+\frac{L\theta}{2d})} \approx \frac{\epsilon_0 L^2}{d}\left(1 - \frac{L\theta}{2d}\right)$$

Alternatively, you can think of this as many small capacitors with parallel plates in parallel (so taking $\theta \to 0$ in the expression from part a). Let the area of these plates be $L\Delta x$. Then, the potential between any of the small plates is (taking a path starting at the center of one plate and going to the other):
Note: Since Δx will be small, the answer we got in part a isn't guaranteed to be valid, but let's assume everything works out and march on

$$\Delta V = -\int_-^+ \vec{E}\cdot d\vec{l} = \int_0^{d(x)} \frac{Q}{\epsilon_0 L\Delta x}dh = \frac{Qd(x)}{\epsilon_0 L\Delta x}$$

Where $d(x)$ is distance between the small plates a distance x from the edge of the total plate (collection of small plates). We see that $d(x)$ is a linear function with $d(0) = d$ and $d(L) = d + L\tan(\theta)$. Thus, $d(x) = d + x\tan(\theta) \approx d + x\theta$.
So the capacitance of a small capacitor ΔC is:

$$\Delta C = \frac{Q}{\Delta V} = \frac{\epsilon_0 L \Delta x}{d(x)}$$

Since the small capacitors are in parallel, the net capacitance is the sum of all of them. When we take the size of the capacitors to be infinitesimally small, we get that:

$$C = \int dC = \int_0^L \frac{\epsilon_0 L}{d + x\theta} dx \approx \int_0^L \frac{\epsilon_0 L}{d}\left(1 - \frac{x\theta}{d}\right) dx = \frac{\epsilon_0 L^2}{d}\left(1 - \frac{L\theta}{2d}\right)$$

The potential energy stored is then:

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2\frac{\epsilon_0 L^2}{d}\left(1 - \frac{L\theta}{2d}\right)} \approx \frac{Q^2 d}{2\epsilon_0 L^2}\left(1 + \frac{L\theta}{2d}\right)$$