Solution to Final Exam, Problem 1

This problem is broken into two parts. In the first part, we compute V_{int} using the adiabatic formula. In the second part, we compute the entropy change in going from V_{int} to V_2.

Part 1:

$P_1 V_1^\gamma = P_2 V_{int}^\gamma$ (4 points)

$\gamma = \frac{d+2}{d} = \frac{7}{5}$ (d=5 since diatomic) (2 points)

$V_{int} = V_1 (\frac{P_1}{P_2})^{5/7}$ (2 points)

$\Delta S = 0$ for adiabtic processes (3 points)

Part 2:

$dS = \frac{dQ}{T}$ (2 points)

$dQ = dE - dW$ (2 points)

$dQ = C_v dT - PdV$ (1 point)

$dS = \frac{C_v dT}{T} - \frac{PdV}{T}$ (1 point)

$dS = \frac{d}{2} nR \frac{nR}{PV} \frac{P}{nR} dV - P \frac{nR}{PV} dV$

$\Delta S = (\frac{d}{2} + 1) nR \ln(\frac{V_2}{V_{int}})$ (1 point)

$\Delta S = \frac{d+2}{2} nR \ln(\frac{V_2}{V_int})$ (or 7 points for writing this)

$\Delta S = \frac{7}{2} nR \ln(\frac{V_2}{V_1} (\frac{P_2}{P_1})^{5/7}$

Solution to Final Exam, Problem 2

We may treat the capacitor in this problem as a sequence of infinitessimal capacitors in parallel. This is justified becasue the field will only be in the z direction, which can be proven rigorously using more complicated techniques, but is a sensible approximation which just happens to be exact except for fringing effects. For an infinitesimal capacitor dC of length dx,

$$dC = \frac{\epsilon_0 A}{dL} K(x) dx$$

$$C = \int C(x) dx = \int_0^L \frac{\epsilon_0 A}{dL}(1 + \frac{K_0 x}{L}) dx$$

We find at once: $\boxed{C = \frac{\epsilon_0 A}{d}(1 + \frac{K_0}{2})}$. Note we recover the usual result when $K_0 = 0$.

Rubric: 10 points to get the idea and equation for an infinitesimal capacitor of length dx, 5 points for the idea to sum infitesimal capacitors over the length L, and 5 points for carrying out the integral.

Solution to Final Exam, Problem 3

The crossbar divides the rectangular loop into two sub-loops. In each sub-loop, the magnetic flux is chaning and so there will be an induced voltage and current. Both of these currents will pass through the crossbar and feel a force due to the magnetic field. In order to have the crossbar move at a constant velocity, the net force acting on it must be zero, and so the external force that needs to be applied to the bar is the opposite of the magnetic force.

To find this force, first conisder the current in each sub-loop seperatley. For the left sub-loop, the magnetic flux is $\Phi_L = Bwx$ and so the magnitude of the induced voltage is $V_L = |\dot{\Phi}_L| = Bw\dot{x} = Bwv$. The current flowing through the left sub-loop is then, by Ohm's law:

$$I_L = \frac{Bwv}{R_L}$$

where R_L is the total resistance of the wires that comprise the left sub-loop. The resistance of a wire with resistivity ρ, cross-section A, and length d is $R = \rho d/A$. In this case, the total length of wire with non-zero resistivity is $w + 2x$, and so the resistance is $R = \rho\,(2x + w)\,/A$. Thus, the current is:

$$I_L = \frac{BwvA}{\rho}\frac{1}{w + 2x}$$

The flux through the left sub-loop is increasing out of the page, and so by Lenz's law this current will flow clockwise around the left sub-loop.

Now, consider the right sub-loop. The flux is $\Phi_R = Bw\,(l - x)$ and so the magnitude of the induced voltage is again $V_R = |\dot{\Phi}_R| = Bwv$. The lenght of wire is now $d = w + 2(l - x)$ and so the induced current has a magnitude of:

$$I_R = \frac{BwvA}{\rho}\frac{1}{w + 2l - 2x}$$

In this case, the flux is decreasing out of the page, and so the induced current will flow counter-clockwise.

Since the left current flows clockwise and the right current counter-clockwise, in the crossbar both currents are flowing in the downward direction. The total current flowing downard is the sum of the current due to each loop:

$$I = \frac{BwvA}{\rho}\left(\frac{1}{w + 2x} + \frac{1}{w + 2l - 2x}\right)$$

This current is perpendicular to the magnetic field, and so magnetic force on it has a magnidute of $F = IwB$. The direction is given by the right hand rule, and points to the left. Therefore, using the above expression for I, the force we need to apply to maintain constant velocity has a magnidute:

$$\begin{aligned} F &= \frac{B^2w^2vA}{\rho}\left(\frac{1}{w + 2x} + \frac{1}{w + 2l - 2x}\right) \\ &= \frac{2B^2w^2vA}{\rho}\cdot\frac{w + l}{(w + 2x)\,(w + 2l - 2x)} \end{aligned}$$

and points to the right.

Solution to Final Exam, Problem 4

There is a net force on the current loop because the magnetic field is not uniform. We can see from symmetry that the forces on the top and bottom legs of the loop cancel out, so we only need to worry about the side legs. If the left leg is at position x_1 and the right leg is at $x_2 = x_1 + s$, the forces are

$$\mathbf{F}_{M,l} = s\mathbf{I} \times \mathbf{B} = sIB(x_1) = sIB_0 \frac{x1}{a} \hat{\mathbf{x}} \tag{1}$$

$$\mathbf{F}_{M,r} = s(-) \times \mathbf{B} = -sIB(x_2) = -sIB_0 \frac{(x_1 + s)}{a} \hat{\mathbf{x}} \tag{2}$$

since the current runs in the opposite direction on the opposite leg. The net force on the loop is thus

$$\mathbf{F}_M = \mathbf{F}_{M,l} + \mathbf{F}_{M,r} = -\frac{IB_0 s^2}{a} \hat{\mathbf{x}}. \tag{3}$$

Recall from previous physics classes that the force of static friction satisfies

$$F_s \leq \mu_s F_N \tag{4}$$

where F_N is the normal force at the interface where the friction acts, such that the friction force balances the other horizontal forces on the object provided the coefficient of static friction μ_s is sufficiently great. The normal force balances gravity here, so the minimum coefficient to keep the loop stationary satisfies

$$F_f = \mu_s mg = F_M. \tag{5}$$

A tiny bit of algebra yields

$$\mu_s = \frac{IB_0 s^2}{amg} \tag{6}$$

as the minimum coefficient of static friction to keep the loop stationary.

Solution to Final Exam, Problem 5

To solve the circuit, we need to use Kirchoff's rules. We cannot trivially use an equivalent resistance in this case, because it is not just a single voltage source system, where all of the resistors can be said to either be parallel or series to one another. We will name the current going in the conventional direction through $2V$ as I_2 and the current going through V as I_1. The Kirchoffs rules are as follow:

$$2V = (I_1 + I_2)(3R) + I_2(R) \tag{1}$$
$$V = (I_1 + I_2)(3R) + I_1(2R) \tag{2}$$

We need to group this system of equations by I_n in order to solve it. We do this as

$$2V = 3RI_1 + 4RI_2 \tag{3}$$
$$V = 5RI_1 + 3RI_2 \tag{4}$$
$$\tag{5}$$

which now is solvable. Making the coefficients of I_1 equal and subtracting the two equations, we get

$$7V = 11RI_2$$

.

Repeating this procedure for I_2 yields

$$2V = -11RI_2$$

.

Thus, we solve the problem as:

$$I_1 + I_2 = (\frac{7}{11} - \frac{2}{11})V/R = \frac{5}{11}V/R$$

Solution to Final Exam, Problem 6

First use Gauss's Law to find magnetic field a distance r from a wire:
$B = \frac{\mu_0 I}{2\pi r}\hat{z}$ (7 points)
This could be written down from the cheat sheet, or points are given for an appropriate derivation.
We need to integrate over the whole plate:
$B = \int \frac{\mu_0 dI}{2\pi r}$ (4 points)
$dI = jdr$ (5 points)
Plugging this in:
$B = \frac{\mu_0}{2\pi}\int_x^{x+w} \frac{jdr}{r} = \frac{\mu_0 j}{2\pi}\ln(1+\frac{w}{x})$
2 points for correct limits, 2 points for the correct answer.
It *is* possible to do this using Biot-Savart law. But this is much harder. Then, 11 points are given for writing the correct integral, 5 points for inserting j correctly, 2 points for correct limits, and 2 points for the correct answer.

Solution to Final Exam, Problem 7

Problem 7

a)

We want to get to a differential equation to solve for Q_C, the charge on the capacitor, since the probelm asks about the capacitor discharging. Take two loops: one from the capacitor to L_1 and back, and the other around the outside, from the capcitor to L_2 and back.

At the top junction, let i be the current flowing rightward into the junction. Let i_1 be the current flowing downward out of the junction, across L_1 and let i_2 be the current flowing rightward out of the junction across L_2.

Apply the loop rule:

$$V_C - V_{L_1} = 0$$

$$V_C - V_{L_2} = 0$$

Where V_C is the voltage across the capacitor. Apply the junction rule:

$$i = i_1 + i_2$$

Since for any inductior, $V_L = L\frac{di}{dt}$, the loop rule equations become:

$$V_C = L_1 \frac{di_1}{dt}$$

$$V_C = L_2 \frac{di_2}{dt}$$

Since the capacitor is discharging, $i = -\frac{dQ_C}{dt}$. So the junction rule becomes:

$$i = i_1 + i_2 = -\frac{dQ_C}{dt}$$

Now take the derivative of the new junction rule:

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} = -\frac{d^2Q_C}{dt^2}$$

Solving for $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$ from the loop rule and plugging into the junction rule:

$$\frac{V_C}{L_1} + \frac{V_C}{L_2} = -\frac{d^2Q_C}{dt^2}$$

And since $Q_C = CV_C$, we plug into the equation above:

$$\frac{Q_C}{CL_1} + \frac{Q_C}{CL_2} = -\frac{d^2Q_C}{dt^2}$$

Or, with some algebra:

$$-(\frac{L_1 + L_2}{CL_2L_1})Q_C = \frac{d^2Q_C}{dt^2}$$

The solution to this differential equation has the form $Q_C = Acos(\omega t) + Bsin(\omega t)$ where

$$\omega = \sqrt{\frac{L_1 + L_2}{CL_1 L_2}}$$

Apply initial conditions:

$$Q_C(t = 0) = Q_0 = A$$

$$\frac{dQ_C}{dt}(t = 0) = 0 = B$$

So $Q_C(t) = Q_0 cos(\omega t)$ where ω is defined above.

$Q_C(t) = 0$ is the condition to discharge. This is an oscillating circuit, so the shortest time, T, that it takes for the capacitor to fully discharge is when

$$\omega T = \frac{\pi}{2}$$

solving for T:

$$T = \frac{\pi}{2\omega} = \frac{\pi}{2}\sqrt{\frac{CL_1 L_2}{L_1 + L_2}}$$

b)

We already know that $i = -\frac{dQ_C}{dt}$; this same i is the current across the capactior. So we can use our answer from part a) and take the derivative:

$$i(t) = \omega Q_0 sin(\omega t)$$

At $t = T$, this becomes:

$$i(t = T) = \omega Q_0 sin(\omega \frac{\pi}{2\omega}) = \omega Q_0 sin(\frac{\pi}{2}) = \omega Q_0$$

which is the maximum current! So we note that this makes sense, because when the charge on the capcitor is at a minimum, the current should be at its maximum value.