ME185 - Introduction to Continuum Mechanics

Midterm Exam II

Problem 1. (10+10 points)

(a) Let ρ be the mass density, **v** be the velocity vector, **T** be the Cauchy stress tensor, and **b** be the body force per unit mass for a continuum undergoing a deformation. Write the component forms of the local spatial equations for conservation of mass and balance of linear momentum involving the material time derivatives of the relevant quantities.

(b) Use the above equations to show that

$$\frac{\partial(\rho v_i)}{\partial t} = \rho b_i + \frac{\partial}{\partial x_j}(T_{ij} - \rho v_i v_j) \qquad (1)$$

Problem 2. (5+5+10+5 points)

The components with respect to the fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Cauchy stress tensor **T** at the origin of the spatial coordinate system at some instant in time is found to be,

$$[\mathbf{T}] = \tau \begin{bmatrix} 5 & 0 & 2 \\ 0 & 7 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \qquad (2)$$

where τ is a positive constant.

(a) Determine the traction vector **t** acting on the plane

$$x_1 + x_2 + x_3 = 0. \qquad (3)$$

(b) Determine the normal component N and shear component S of the traction determined in part (a). Recall that

$$\mathbf{t} = N\mathbf{n} + S\mathbf{s}. \qquad (4)$$

where **n** is the normal vector to the plane given in Eq. (3), **s** lies in that plane, and $S \geq 0$.

(c) Determine the maximum and minimum principal stresses, and the planes on which they act. Each plane can be specified either through its defining equation (as in Eq. (3)) or through its unit normal vector.

(d) For the stress given by Eq. (2), are there planes containing this point on which the normal component of the traction vector vanishes? Justify your answer.

<u>Problem 3.</u> (15+10 points)

Consider the motion of a body for which the deformation gradient at some material point X is given as **F**. Let **T** denote the Cauchy stress tensor, and let **S** denote the second (symmetric) Piola-Kirchhoff stress tensor. Recall that

$$\mathbf{T} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T, \tag{5}$$

where J is the Jacobian of the deformation.

(a) The Truesdell stress rate $\overset{\triangleright}{\mathbf{T}}$ is defined as

$$\overset{\triangleright}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T + \mathbf{T}\,tr\mathbf{D}, \tag{6}$$

where **L** is the spatial velocity gradient of the deformation and **D** is the stretching (the symmetric part of **L**). Use Eqs. (5) and (6) to show that

$$\overset{\triangleright}{\mathbf{T}} = \frac{1}{J}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^T. \tag{7}$$

(b) Use Eq. (7) and the fact that **S** is invariant under a superposed rigid-body motion,

$$\mathbf{x}^+ = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \tag{8}$$

to show that $\overset{\triangleright}{\mathbf{T}}$ is an objective Eulerian tensor in the sense that

$$\overset{\triangleright}{\mathbf{T}}{}^+ = \mathbf{Q}\overset{\triangleright}{\mathbf{T}}\mathbf{Q}^T. \tag{9}$$

Problem 4. (10+20 points)

An initially homogeneous body, with mass density ρ_o in the reference configuration, is subject to surface tractions, but no body forces. The resulting deformation is found to be

$$\begin{aligned} x_1 &= \chi_1(\mathbf{X},t) = X_1 + \alpha^2 t^2 X_2 \\ x_2 &= \chi_2(\mathbf{X},t) = X_2(1+\alpha t) \\ x_3 &= \chi_3(\mathbf{X},t) = X_3 \end{aligned} \tag{10}$$

where α is a positive constant.

(a) Determine the mass density as a function of spatial position and time.

(b) The first Piola-Kirchhoff stress tensor for this deformation may be written in component form as

$$[\mathbf{P}] = \begin{bmatrix} \beta(t) X_2 & \gamma(t) X_2^2 & 0 \\ \delta(t) X_2^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{11}$$

where β, γ and δ are scalar functions of time. Use balance of linear momentum and balance of angular momentum to obtain restrictions on the functions $\beta(t)$, $\gamma(t)$ and $\delta(t)$ in terms of the prescribed constant α.

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Problem 1. (30 Points)

Consider the motion of a body for which the deformation is given by the mapping $\chi(\mathbf{X},t)$. Suppose that the integral equation

$$\frac{d}{dt}\int_{P}\varphi d\mathrm{v} = \int_{P} r d\mathrm{v} + \int_{\partial P}\mathbf{h}\cdot\mathbf{n}d\mathrm{a} \qquad (1)$$

holds for all parts P with boundary ∂P. In this equation, φ and r are scalar functions of $\mathbf{x}$ and t, while $\mathbf{h}$ is a vector function of the same variables. The vector $\mathbf{n}$ is the outward unit normal to ∂P.

Use appropriate integral theorems to derive a local partial differential equation involving φ, r and $\mathbf{h}$. Be clear in your identification and application of the various integral theorems, and be sure that your final equation involves the *partial derivatives* of the relevant quantities with respect to $\mathbf{x}$ and t.

Problem 2 (30 Points)

Recall that under a superposed rigid-body motion,

$$\mathbf{x}^{+} = \mathbf{Q}(t)\mathbf{x} + \mathbf{a}(t)\,, \qquad (2)$$

where $\mathbf{Q}(t)$ is an arbitrary proper orthogonal tensor and $\mathbf{a}(t)$ is an arbitrary vector, the Cauchy stress tensor transforms as

$$\mathbf{T}^{+} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{T}\,. \qquad (3)$$

This implies that $\mathbf{T}$ is an objective Eulerian tensor. Also recall that the rotation tensor $\mathbf{R}$ obtained from the polar decomposition theorem, $\mathbf{F} = \mathbf{R}\mathbf{U}$, transforms as

$$\mathbf{R}^{+} = \mathbf{Q}\mathbf{R}\,. \qquad (4)$$

Let the *Green-McInnis* rate of Cauchy stress tensor be defined as

$$\overset{\Diamond}{\mathbf{T}} = \dot{\mathbf{T}} - \dot{\mathbf{R}}\mathbf{R}^{T}\mathbf{T} + \mathbf{T}\dot{\mathbf{R}}\mathbf{R}^{T}\,. \qquad (5)$$

Show that $\overset{\Diamond}{\mathbf{T}}$ is an objective Eulerian tensor.

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Problem 3 (40 Points)

Let body $\mathcal{B}$ be at rest in the current configuration κ and occupy a region $\mathcal{R}$, given as a parallelepiped centered at the origin and having edges of length $2l$, $2w$ and $2h$,

$$\mathsf{R} = \{(x_1, x_2, x_3) \in \mathsf{E}^3 \mid \ |x_1| \le l,\ |x_2| \le w,\ |x_3| \le h\}. \qquad (6)$$

Let the components of the Cauchy stress tensor with respect to fixed orthonormal basis $\{\mathbf{e}_i\}$ be of the form

$$[\mathbf{T}]_{\mathbf{e}_m \otimes \mathbf{e}_n} = A \begin{bmatrix} a x_1^2 & 0 & c x_2 - d x_1^2 x_2 \\ 0 & b x_2^2 & e x_1 - f x_2^2 x_1 \\ c x_2 + x_1^2 x_2 & e x_1 - f x_2^2 x_1 & 0 \end{bmatrix}_{\mathbf{e}_m \otimes \mathbf{e}_n}. \qquad (7)$$

(a) Determine the coefficients a, b, c, d, e and f using the balance of linear and angular momentum and the conditions that:

(i) The motion exists in the absence of body forces
(ii) The faces $x_1 = \pm l$ and $x_2 = \pm w$ are free of tractions

(b) Determine the traction vector acting on the face $x_3 = h$ as a function of x_1 and x_2.

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Problem 1. (20 points)

Consider a body for which the Cauchy stress is given as

$$T_{ij} = -p\delta_{ij}\,. \tag{1}$$

Show that the stress power for this body may be expressed as

$$T_{ij}D_{ij} = p\frac{\dot{\rho}}{\rho}\,. \tag{2}$$

Problem 2 (20 Points)

The Cauchy stress tensor at some point in a body is given in matrix form as

$$\mathbf{T} = \tau\begin{bmatrix} a & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{\mathbf{e}_i \otimes \mathbf{e}_j}\,, \tag{3}$$

where τ and a are constants. For some value of the constant a, there is a plane at this point that is traction free. Determine this value of a and the associated unit vector $\mathbf{n}$ which is normal to the traction free plane.

Problem 3 (30 Points)

Let $\mathbf{a}$ represent the acceleration of a material point, $\mathbf{a} = \dot{\mathbf{v}} = a_i\mathbf{e}_i$, and let $\mathbf{A}$ represent the spatial acceleration gradient tensor, defined with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{A} = \frac{\partial a_i}{\partial x_j}\;\mathbf{e}_i \otimes \mathbf{e}_j\,. \tag{4}$$

(a) Show that

$$\ddot{\mathbf{F}} = \mathbf{A}\mathbf{F}\,. \tag{5}$$

(b) Also show that

$$\dot{\overline{(\mathbf{F}^{-1})}} = -\mathbf{F}^{-1}\mathbf{L} \tag{6}$$

(c) Use the above results, along with the definition of **L**, to show that

$$\dot{\mathbf{L}} = \mathbf{A} - \mathbf{L}^2 . \tag{7}$$

Problem 4 (30 Points)

The answers to the following questions are of the "always, sometimes, never" variety. You are asked to state the conditions under which each integral statement is true. It may be that the statement is always true (no conditions must be placed on any of the quantities involved), never true (no conditions on the quantities will make it so) or sometimes true (you must state the specific conditions required).

(a) Consider the integral statement,

$$\int_R \mathbf{r}(\mathbf{x})d\mathrm{v} = 0 . \tag{8}$$

where **r** is a vector. Under what conditions can one conclude that **r**(**x**) = 0?

(b) Consider the integral statement

$$\frac{d}{dt}\int_{\mathrm{P}} \tilde{\varphi}(\mathbf{x},t)\mathrm{dv} = \int_{\mathrm{P}} \left(\frac{\partial \tilde{\varphi}}{\partial t} + \tilde{\varphi}\frac{\partial \tilde{\mathrm{v}}_{\mathrm{i}}}{\partial x_{\mathrm{i}}} \right)\mathrm{dv} . \tag{9}$$

where $\tilde{\varphi}$ is a scalar, $\tilde{v}_i$ are the spatial components of the velocity vector and P is an arbitrary part of a continuous body. Under what conditions does Eq. (9) hold?

(c) Consider the integral statement,

$$\int_{\partial P} \mathbf{g} \cdot \mathbf{n} d\mathrm{a} = 0 . \tag{10}$$

where **g** is a vector and **n** is the unit normal vector to the bounding surface ∂P. Under what conditions does Eq. (10) hold?