ME185
Introduction to Continuum Mechanics

The attached pages contain four previous midterm exams for this course. Each midterm consists of two pages. As you may notice, many of the problems are similar to those found in the homework. Some homework problems are identical to exam problems and some of the exam problems may have been repeated (as I like to reuse problems that are particularly nice in some aspect or another).

The midterm this semester will follow this pattern. In preparing for the exam, you should know how to solve all of the assigned homework problems for the semester, along with those given here.

# ME185 - Introduction to Continuum Mechanics

## Midterm Exam I

Please show your work in answering these three problems. Begin each new problem on a separate sheet and group the sheets for each problem separately. There will be three envelopes in which to put your three problem solutions.

**Problem 1.** (40 points)

Consider a deformation defined by the mapping

$$\begin{aligned} x_1 &= \chi_1(X_A,t) = (1+at^2)X_1 \\ x_2 &= \chi_2(X_A,t) = btX_1 + X_2 \\ x_3 &= \chi_3(X_A,t) = e^{-ct}X_3 \end{aligned} \quad , \tag{1}$$

where $a$, $b$ and $c$ are constants.

(a) What restrictions, if any, must be placed on the constants $a$, $b$ and $c$ in order for this mapping to be admissible for any time $t \geq 0$?

(b) Determine the spatial components of the velocity and acceleration fields for this motion.

(c) Determine the conditions under which this motion:
   (i) Is isochoric
   (ii) Has a steady velocity field

(d) Compute the stretch of a line element whose unit normal at time $t = 0$ is $\mathbf{M} = (\mathbf{E}_2 + \mathbf{E}_3)/\sqrt{2}$

(e) Sketch the path line for a particle starting at $\mathbf{X} = \mathbf{E}_1 + \mathbf{E}_2$ for the case when $a = 1$, $b = 1$ and $c = 0$.

**Problem 2** (35 Points)

Consider a velocity field for which the spatial components with respect to the basis $\{\mathbf{e}_i\}$ are

$$\begin{aligned} \tilde{v}_1 &= (1\text{-}t)x_1 + tx_2 \\ \tilde{v}_2 &= tx_2 \qquad , \\ \tilde{v}_3 &= x_3 \end{aligned} \tag{2}$$

and a scalar field $\phi$, given as

$$\phi = x_1 x_3 e^{-t}. \tag{3}$$

(a) Determine the curl of the velocity gradient (giving the axial vector associated with the spin or vorticity tensor);
(b) Determine the spatial components of the acceleration vector for this motion;
(c) Determine the gradient of $\phi$ with respect to the current coordinates $\mathbf{x}$;
(d) Determine the material time derivative of $\phi$.

**Problem 3.** (25 points)

Consider the motion given by the mapping

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + \gamma t\, X_2 \\ x_2 &= \chi_2(X_A, t) = -\gamma\, t\, X_1 + X_2 \\ x_3 &= \chi_3(X_A, t) = X_3, \end{aligned} \tag{4}$$

where $\gamma$ is a constant. Also consider the proper orthogonal tensor $\mathbf{Q}$, given in matrix form as

$$[\mathbf{Q}] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Can $\mathbf{Q}$ ever represent the rotation tensor associated with the polar decomposition of this motion, and if so, under what circumstances?

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## Midterm Exam I

**Problem 1.** (50 Points)

A homogeneous motion of a body is specified by the mapping

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + atX_2 \\ x_2 &= \chi_2(X_A, t) = -atX_1 + X_2 \\ x_3 &= \chi_3(X_A, t) = X_3 \end{aligned} \tag{1}$$

where $a$ is a constant.

(a) Determine the components of the deformation gradient **F** and verify that the motion given by Eq. (1) is invertible at all times.

(b) Determine the components of the strain Lagrange tensor **E**.

(c) Determine the components of the velocity vector **v** in both the referential and spatial descriptions.

(d) Let a scalar function $\phi$ be defined as

$$\phi = \tilde{\phi}(x_i, t) = x_1 + bt^2 x_2 . \tag{2}$$

Determine the material time derivative of $\phi$.

(e) Determine the components of the rotation tensor **R** and the stretch tensor **U** associated with the polar decomposition **F = RU**.

(f) Determine the stretch of a line element whose direction in the reference configuration is $\mathbf{M} = (\mathbf{E}_1 + \mathbf{E}_2)/\sqrt{2}$ .

(g) Sketch the path line for a particle that at time $t = 0$ occupies a point with position vector $\mathbf{X} = \mathbf{E}_1 + \mathbf{E}_2$ . For this part of the problem, let $a = 1$.

**Problem 2** (30 Points)

Let **a** represent the acceleration of a material point, $\mathbf{a} = \dot{\mathbf{v}} = a_i \mathbf{e}_i$ , and let **A** represent the spatial acceleration gradient tensor, defined with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathbf{A} = \frac{\partial a_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j . \tag{3}$$

(a) Show that

$$\ddot{\mathbf{F}} = \mathbf{A}\mathbf{F} . \tag{4}$$

(b) Also show that

$$\overline{(\mathbf{F}^{-1})}^{\boldsymbol{\cdot}} = -\mathbf{F}^{-1}\mathbf{L} \tag{5}$$

(c) Use the above results, along with the definition of **L**, to show that

$$\dot{\mathbf{L}} = \mathbf{A} - \mathbf{L}^2 . \tag{6}$$

**Problem 3** (20 Points)

Consider a velocity field for which the spatial components with respect to the basis $\{\mathbf{e}_i\}$ are

$$\begin{aligned} \tilde{v}_1 &= (1\text{-}t)x_1 + 2tx_2 \\ \tilde{v}_2 &= (1+t)x_2 - 2tx_3 \, . \\ \tilde{v}_3 &= x_3 \end{aligned} \tag{7}$$

(a) Determine the spatial components of the acceleration vector for this motion.

(b) Determine the rate of stretch **D** and the vorticity **W** for this motion.

(c) Determine if this motion:
 (i) Is isochoric (assume that J=1 at $t$=0);
 (ii) Has a steady velocity field.

# ME185 - Introduction to Continuum Mechanics

## Midterm Exam I

**Problem 1.** (20 points)

Consider a deformation defined by the mapping

$$x_1 = \chi_1(X_A,t) = e^{-\eta t} X_1$$
$$x_2 = \chi_2(X_A,t) = e^{-\eta t} X_2 \qquad , \qquad (1)$$
$$x_3 = \chi_3(X_A,t) = X_3$$

where $\eta$ is a positive constant.

(a) Determine the spatial components of the velocity and acceleration fields for this motion.

(b) Determine the Eulerian strain tensor **e**.

(c) Determine the conditions under which this motion:
(i) Is isochoric
(ii) Has a steady velocity field

**Problem 2.** (25 points)

Consider a velocity field for which the spatial components with respect to the basis $\{\mathbf{e}_i\}$ are

$$\tilde{v}_1 = a x_1 x_2 e^{-\eta t}$$
$$\tilde{v}_2 = -a x_2^2 e^{-\eta t} \; , \qquad (2)$$
$$\tilde{v}_3 = b e^{-\eta t}$$

where $a$, $b$ and $\eta$ are positive constants.

(a) Determine the spatial components of the acceleration for this motion.

(b) Determine the rate of stretch **D** and the vorticity **W** for this motion.

(c) Determine the conditions under which this motion:
(i) Is isochoric (assume that J=1 at $t$=0);
(ii) Has a steady velocity field.

**<u>Problem 3.</u>** (25 points)

Consider a deformation for which:

$\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_3$ are the principal directions (eigenvectors) of the stretch tensor **U.**
The eigenvalues of **U** are 1, 2, and ½ in the $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_3$ directions.
The axis of rotation **p** for the rotation tensor **R** is $(\mathbf{E}_1 + \mathbf{E}_2)/\sqrt{2}$.
The angle of rotation about **p** is $\pi/2$.

(a) Express each of the following tensors with respect to the basis indicated:
The right stretch tensor **U** with respect to $\mathbf{E}_A \otimes \mathbf{E}_B$;
The rotation tensor **R** with respect to $\mathbf{e}_i \otimes \mathbf{E}_A$
The deformation gradient tensor **F** with respect to $\mathbf{e}_i \otimes \mathbf{E}_A$
The left Cauchy-Green stretch tensor **B** with respect to $\mathbf{e}_i \otimes \mathbf{e}_j$

(b) Determine the stretch of a line element which is parallel to **p** in the reference configuration.

**<u>Problem 4.</u>** (30 points)

Consider two material line elements $d\mathbf{X}_{(1)}$ and $d\mathbf{X}_{(2)}$ in the reference configuration $\kappa_o$, which deform into line elements $d\mathbf{x}_{(1)}$ and $d\mathbf{x}_{(2)}$ in the current configuration $\kappa$ through the deformation gradient **F**. Let $\mathbf{M}_{(\alpha)}$ and $\mathbf{m}_{(\alpha)}$ ($\alpha$ = 1,2) be the unit vectors in the directions of $d\mathbf{X}_{(\alpha)}$ and $d\mathbf{x}_{(\alpha)}$, respectively, and let $\lambda_{(\alpha)}$ denote the stretch of these line elements. Finally, let $\theta$ denote the angle between $\mathbf{m}_{(1)}$ and $\mathbf{m}_{(2)}$ in $\kappa$.

(a) Show that

$$\lambda_{(1)}\lambda_{(2)}\cos\theta = \mathbf{M}_{(1)} \cdot \mathbf{C}\mathbf{M}_{(2)}. \tag{3}$$

where **C** is the right Cauchy-Green stretch tensor.

(b) Show that for $\theta \neq 0$, the material time derivative of $\theta$ is given as

$$\dot{\theta} = \frac{1}{\sin\theta}\left[\mathbf{m}_{(1)} \cdot \mathbf{D}(\mathbf{m}_{(1)}\cos\theta - \mathbf{m}_{(2)}) + \mathbf{m}_{(2)} \cdot \mathbf{D}(\mathbf{m}_{(2)}\cos\theta - \mathbf{m}_{(1)})\right]. \tag{4}$$

where **D** is the rate of stretch tensor.

(c) The above equation is undefined when $\theta = 0$. Obtain an expression for $\dot{\theta}$ in this case (i.e., when $\theta = 0$).

# ME185 - Introduction to Continuum Mechanics

## Midterm Exam I

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**Problem 1.** (20 points)

Recall that the polar decomposition of the deformation gradient tensor **F** is written as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}\,. \tag{1}$$

where **R** is the proper orthogonal rotation tensor, and **U** and **V** are the symmetric, positive definite stretch tensors. Let **Ω** be defined as

$$\Omega = \dot{\mathbf{R}}\mathbf{R}^T\,. \tag{2}$$

Prove that **Ω** is skew-symmetric.

**Problem 2.** (5+5+10+10 points)

A homogeneous motion of a deformable body is defined as

$$\begin{aligned} x_1 &= \chi_1(\mathbf{X},t) = \mathrm{X}_1 \cos t + X_2 \sin t \\ x_2 &= \chi_2(\mathbf{X},t) = \text{-}\mathrm{X}_1 \sin t + X_2 \cos t \\ x_3 &= \chi_3(\mathbf{X},t) = \mathrm{X}_3 e^t \end{aligned} \tag{3}$$

(a) Determine the components of the deformation gradient tensor **F**.

(b) Determine the components of the Lagrange strain tensor **E**.

(c) Determine the components of the rotation tensor **R** and the stretch tensor **U** associated with the polar decomposition of **F** as in Eq. (1).

(d) Determine the components of the rate of deformation tensor **D** and the vorticity tensor **W**, recalling that **D** and **W** are, respectively, the symmetric and skew-symmetric parts of the spatial velocity gradient tensor **L**.

**Problem 3.** (20 points)

For a plane motion of a body, the rate of deformation tensor **D** and vorticity tensor **W** may be written in component form as,

$$[\mathbf{D}] = \begin{bmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{W}] = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

where *a, b, c* and *w* may be functions of both spatial position **x** and time *t*. Use this to show that, for such a plane motion,

$$\mathbf{WD} + \mathbf{DW} = (\text{div}\,\mathbf{v})\mathbf{W}, \tag{5}$$

where **v** is the velocity vector for the motion.

**<u>Problem 4.</u>** (10+10+10 points)

A motion of a deformable body is defined as

$$\begin{aligned} x_1 &= \chi_1(\mathbf{X},t) = X_1 + t \\ x_2 &= \chi_2(\mathbf{X},t) = X_2(2+\cos t) \\ x_3 &= \chi_3(\mathbf{X},t) = X_3 e^t \end{aligned} \tag{6}$$

(a) Determine the spatial components of the velocity vector **v**.

(b) Determine the components of the spatial velocity gradient tensor **L**.

(c) Let $\psi$ be the scalar function

$$\psi = e^{-t} + \frac{x_1}{x_3}. \tag{7}$$

Determine the material time derivative $\dot{\psi}$ as a function of **x** and *t*.