Solutions to Midterm 2

Problem 1. The center O of the rolling cylinder moves in a circle about B. Thus

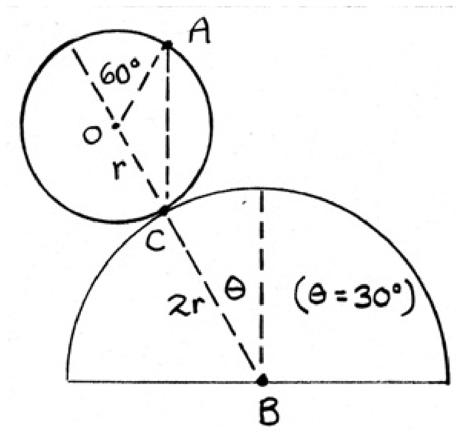
$$v_O = OB\dot{\theta} = 3r\dot{\theta}$$

The contact point C is the instantaneous center of zero velocity of the rolling cylinder.

$$\omega = \frac{v_O}{r} = \frac{3r\dot{\theta}}{r} = 3\dot{\theta}$$

Any line on the rolling cylinder has angular velocity ω. Since A moves in a circle relative to C,

$$v_A = AC\omega_{AC} = 2r\cos 30°\omega = 3\sqrt{3}r\dot{\theta}$$



Problem 2. Let v_0 be the velocity of carriage after impact. For the system of sphere and carriage,

$$\Delta G_x = 0 \qquad \Rightarrow \qquad 2(10) = 2(-v'\cos\theta) + 10v_0 \qquad (1)$$

In the oblique central impact, momentum of the sphere is conserved in *t*-direction. Thus

$$\Delta G_t = 0 \qquad \Rightarrow \qquad 2(10\sin 30°) = 2v'\sin(\theta - 30°) \qquad (2)$$

Restitution in *n*-direction gives

$$0.6 = \frac{v_0\cos 30° + v'\cos(\theta - 30°)}{10\cos 30°} \qquad (3)$$

There are 3 unknowns v', θ and v_0 in 3 equations. Upon solution,

$$v' = 6.04\,\text{m/s}$$
$$\theta = 85.9°$$
$$v_0 = 2.087\,\text{m/s}$$

For the carriage after impact,

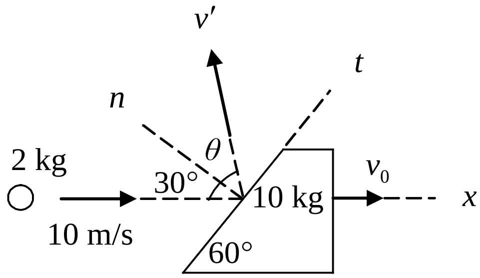
$$\Delta T + \Delta V_e = 0$$

$$\Rightarrow \qquad -\frac{1}{2}(10)v_0^2 + \frac{1}{2}(1600)\delta^2 = 0$$

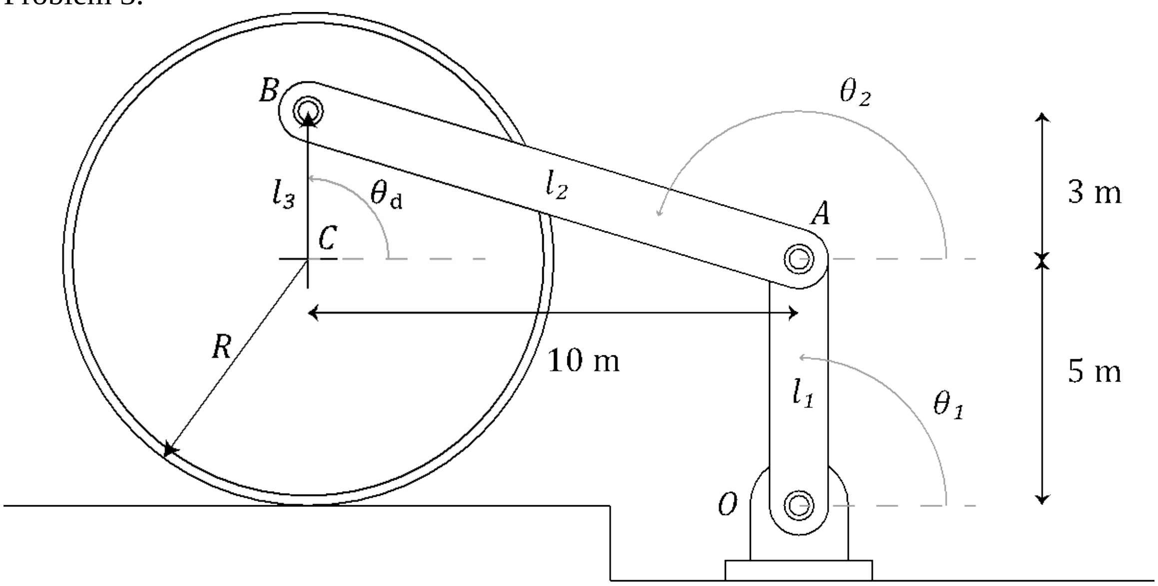
$$\Rightarrow \qquad \delta = \pm 0.165$$

Take the positive root to write

$$\delta = 165\,\text{mm}$$



Problem 3.



The wheel shown rolls without slipping on the horizontal surface. At the instant shown, it has an angular velocity of $\dot{\theta}_d$ and an angular acceleration of $\ddot{\theta}_d$.

a) Determine the single **vector** equation and the corresponding two **scalar** equations for $\dot{\theta}_1$ and $\dot{\theta}_2$, in terms of general $l_1, l_2, l_3, R, \theta_1, \theta_2, \dot{\theta}_d$ and $\ddot{\theta}_d$ (40%)

First compare two vector relationships for the position C:

$\boldsymbol{r_C} = \boldsymbol{r_A} + \boldsymbol{r_{B/A}} + \boldsymbol{r_{C/B}} \rightarrow \boldsymbol{r_C} = l_1\boldsymbol{e_{r1}} + l_2\boldsymbol{e_{r2}} - l_3\boldsymbol{e_{rd}}$

where θ_1, θ_2 and their corresponding basis have defined in the standard, counterclockwise fashion. Next, differentiate:

$\boldsymbol{v_C} = l_1\dot{\theta}_1\boldsymbol{e_{\theta 1}} + l_2\dot{\theta}_2\boldsymbol{e_{\theta 2}} - l_3\dot{\theta}_d\boldsymbol{e_{\theta d}}$

and, recognizing that we can determine $\boldsymbol{v_c}$ from the rolling condition:

$\boldsymbol{v_c} = -\boldsymbol{\omega_d} \times (\boldsymbol{r_p} - \boldsymbol{r_c}) = -\dot{\theta}_d\boldsymbol{E_z} \times -R\boldsymbol{E_y} \rightarrow \boldsymbol{v_c} = -R\dot{\theta}_d\boldsymbol{E_x}$

we have:

$-R\dot{\theta}_d\boldsymbol{E_x} + l_3\dot{\theta}_d\boldsymbol{e_{\theta d}} = l_1\dot{\theta}_1\boldsymbol{e_{\theta 1}} + l_2\dot{\theta}_2\boldsymbol{e_{\theta 2}}$

Dotting with the x and y basis vectors yields, in matrix form,

$$\begin{bmatrix} -l_1\sin(\theta_1) & -l_2\sin(\theta_2) \\ l_1\cos(\theta_1) & l_2\cos(\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -\dot{\theta}_d\{R + l_3\sin(\theta_d)\} \\ \dot{\theta}_d l_3\cos(\theta_d) \end{bmatrix}$$

For the numerical values of the constants as given with $\dot{\theta}_d$ = -6 rad/sec and $\ddot{\theta}_d$ = -2 rad/sec^2, determine $\dot{\theta}_1$ and $\dot{\theta}_2$ (10%)

$$\begin{bmatrix} -5 & -3 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 6\{5+3\} \\ 0 \end{bmatrix}$$

Solution yields $\dot{\theta}_1$ = -9.6 rad/s and $\dot{\theta}_2$ = 0 rad/sec

b) Determine the single **vector** equation and the corresponding two **scalar** equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$, in terms of general $l_1, l_2, l_3, R, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \dot{\theta}_d$ and $\ddot{\theta}_d$ (45%)

Differentiating yet again, we have:

$$\boldsymbol{a_c} = l_1\left\{\ddot{\theta}_1\boldsymbol{e_{\theta 1}} - \dot{\theta}_1^{\,2}\boldsymbol{e_{r1}}\right\} + l_2\left\{\ddot{\theta}_2\boldsymbol{e_{\theta 2}} - \dot{\theta}_2^{\,2}\boldsymbol{e_{r2}}\right\} - l_3\left\{\ddot{\theta}_d\boldsymbol{e_{\theta d}} - \dot{\theta}_d^{\,2}\boldsymbol{e_{rd}}\right\}$$

but $\boldsymbol{a_c} = -R\ddot{\theta}_d\boldsymbol{E_x}$, so we have:

$$-R\ddot{\theta}_d\boldsymbol{E_x} + l_3\left\{\ddot{\theta}_d\boldsymbol{e_{\theta d}} - \dot{\theta}_d^{\,2}\boldsymbol{e_{rd}}\right\} + l_1\dot{\theta}_1^{\,2}\boldsymbol{e_{r1}} + l_2\dot{\theta}_2^{\,2}\boldsymbol{e_{r2}} = l_1\ddot{\theta}_1\boldsymbol{e_{\theta 1}} + l_2\ddot{\theta}_2\boldsymbol{e_{\theta 2}}$$

Dotting with the x and y basis vectors yields, in matrix form,

$$\begin{bmatrix} -l_1\sin(\theta_1) & -l_2\sin(\theta_2) \\ l_1\cos(\theta_1) & l_2\cos(\theta_2) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\ddot{\theta}_d\{R + l_3\sin(\theta_d)\} - l_3\dot{\theta}_d^{\,2}\cos(\theta_d) + l_1\dot{\theta}_1^{\,2}\cos(\theta_1) + l_2\dot{\theta}_2^{\,2}\cos(\theta_2) \\ \ddot{\theta}_d l_3\cos(\theta_d) - l_3\dot{\theta}_d^{\,2}\sin(\theta_d) + l_1\dot{\theta}_1^{\,2}\sin(\theta_1) + l_2\dot{\theta}_2^{\,2}\sin(\theta_2) \end{bmatrix}$$

For the numerical values of the constants given and the previously determined values of $\dot{\theta}_1$ and $\dot{\theta}_2$, determine $\ddot{\theta}_1$ and $\ddot{\theta}_2$. (5%)

$$\begin{bmatrix} -5 & -3 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 2\{5+3\} - 0 + 0 + 0 \\ 0 - 3(-6)^2 + 5(-9.6)^2 + 3(0)^2 \end{bmatrix}$$

Solution yields $\ddot{\theta}_1$ = 17.97 rad/s^2 and $\ddot{\theta}_2$ = -35.28 rad/s^2