UNIVERSITY OF CALIFORNIA, BERKELEY

DEPARTMENT OF STATISTICS

STAT 134: Concepts of Probability

Spring 2014

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Solution to the Midterm Examination

1. A point (X, Y) is randomly selected from the following finite set of points on the plane

$$\left\{ (x, y) \,\middle|\, 1 \leq y \leq x \leq n \right\}.$$

(a) What are the marginal distributions of X and Y? Is the pair (X, Y) *exchangeable*? Explain your answer. [2 + 1]

First note that the total number of points in the given set is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

The values of the random variable X are $\{1, 2, \cdots, n\}$. Fix $1 \leq x \leq n$, then

$$\mathbf{P}(X = x) = \sum_{y=1}^{x} \mathbf{P}(X = x, Y = y) = \frac{2x}{n(n+1)}.$$

The values of the random variable Y are $\{1, 2, \cdots, n\}$. Fix $1 \leq y \leq n$, then

$$\mathbf{P}(Y = y) = \sum_{x=y}^{n} \mathbf{P}(X = x, Y = y) = \frac{2(n - y + 1)}{n(n+1)}.$$

Note that always $Y \leq X$ and hence the pair (X, Y) is not *exchangeable*.

(b) What is the distribution of the random variable $X + Y$? [5]

Let $Z := X + Y$, then the values of Z are $\{2, 3, \cdots, 2n\}$. Fix $2 \leq z \leq 2n$, then

$$\mathbf{P}(Z = z) = \sum_{x=\lceil \frac{z}{2} \rceil}^{\min(z-1,n)} \mathbf{P}(X = x, Y = y) = \begin{cases} \frac{2\left(z - \lceil \frac{z}{2} \rceil\right)}{n(n+1)} & \text{if } \; 2 \leq z \leq n+1 \\ \\ \frac{2\left(n - \lceil \frac{z}{2} \rceil + 1\right)}{n(n+1)} & \text{if } \; n+1 < z \leq 2n, \end{cases}$$

where $\lceil u \rceil$ means the least integer greater or equal to the real number u.

(c) Find the conditional distribution of X given $X + Y = n$. [4]

Given $[X + Y = n]$ the values of the random variable X are $\left\{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \cdots, n-1\right\}$. Fix $\lceil \frac{n}{2} \rceil \leq x \leq n-1$, then

$$\mathbf{P}\left(X = x \,\middle|\, X + Y = n\right) = \frac{\mathbf{P}\left(X = x, Y = n - x\right)}{\mathbf{P}\left(X + Y = n\right)} = \frac{1}{n - \lceil \frac{n}{2} \rceil}.$$

This $X \,\Big|\, X + Y = n \sim \text{Uniform}\left\{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \cdots, n-1\right\}$.

2. Suppose $N \sim \text{Negative-Binomial}\,(2, p)$ where $0 < p < 1$. Let X be a random variable such that the conditional distribution of X given $N = n$ is Uniform $(\{1, 2, \cdots, n-1\})$. Find the marginal distribution of X. What is the conditional distribution of N given $X = 10$. [8 + 4]

First we note that the values of X are $\{1, 2, \cdots\}$. Now fix $x \geq 1$,

$$\begin{aligned}
\mathbf{P}\left(X = x\right) &= \sum_{n=x+1}^{\infty} \mathbf{P}\left(X = x \,\middle|\, N = n\right) \mathbf{P}\left(N = n\right) \\
&= \sum_{n=x+1}^{\infty} \frac{1}{n-1} (n-1)\, p^2 (1-p)^{n-2} \\
&= p^2 (1-p)^{x-1} \sum_{k=0}^{\infty} (1-p)^k \\
&= p\,(1-p)^{x-1}.
\end{aligned}$$

So we get $X \sim \text{Geometric}\,(p)$.

Now, give $[X = 10]$, the values of N are $\{11, 12, \cdots\}$. Fix $n \geq 11$,

$$\begin{aligned}
\mathbf{P}\left(N = n \,\middle|\, X = 10\right) &= \frac{\mathbf{P}\left(X = 10 \,\middle|\, N = n\right) \mathbf{P}\left(N = n\right)}{\mathbf{P}\left(X = 10\right)} \\
&= p\,(1-p)^{n-11}.
\end{aligned}$$

Thus $N \Big| X = 10 \sim 10 + \text{Geometric}\,(p)$.

3. In a Quidditch trail Harry asked Ginny to try to score a goal with Ron as the keeper. Ginny who is an excellent player has a chance of scoring a goal 90% of the times. The game was stopped as soon as Ginny scored a goal. It took exactly 9 minutes for her to score a goal. Find the expected time Ginny took for each of her tries. Explain the assumptions you are making. [10 + 2]

We assume that each try of Ginny are independent Bernoulli trials with success probability 0.9. We also assume that that each try Ginny takes equal amount of time.

Let X be the number of trials Ginny made, then $X \sim \text{Geometric}\,(0.9)$. So the expected time Ginny took for each of her tries is

$$\begin{aligned}
\mathbf{E}\left[\frac{9}{X}\right] &= \sum_{k=1}^{\infty}\frac{9}{k}\times 0.9\times 0.1^{k-1}\\
&= \frac{9p}{q}\sum_{k=1}^{\infty}\frac{q^k}{k} \qquad\qquad [p=0.9 \text{ and } q=0.1]\\
&= \frac{9p}{q}\sum_{k=1}^{\infty}\int_0^q t^{k-1}\,dt\\
&= \frac{9p}{q}\int_0^q\left(\sum_{k=1}^{\infty}t^{k-1}\right)dt\\
&= \frac{9p}{q}\int_0^q\frac{dt}{1-t}\\
&= -\frac{9p}{q}\log p = 81\log\frac{10}{9}\approx 8.5342.
\end{aligned}$$

4. Ron was given the task of sending out the invitations for the wedding of Ginny and Harry. They wanted to invite 100 of their friends. Unfortunately, Ron did not realized that his charm made the 100 invitation letters to be randomly placed in the 100 addressed envelop. The outcome was a disaster, it may have been that an invitation has gone inside an envelop addressed to a different person, or an envelop containing more than one invitations or none at all. In all such cases the recipients were either confused or upset and they did not show up. Only the invitees who received the invitations inside a correctly addressed envelop came for the wedding. Let X be the total number of people who came for the wedding.

   (a) Find $\mathbf{P}(X=100)$ and $\mathbf{P}(X=0)$. [3 + 4]

   Let $A_i :=$ i^{th} invitation is the only one which has gone inside the i^{th} envelop. Then by definition

$$X=\sum_{i=1}^{100}\mathbf{1}_{A_i}.$$

   Thus,

$$\mathbf{P}(X=100)=\mathbf{P}\left(\bigcap_{i=1}^{100}A_i\right)=\frac{1}{100^{100}}.$$

   Further,

$$\begin{aligned}
\mathbf{P}(X=0) &= 1-\mathbf{P}(X\geq 1)\\
&= 1-\mathbf{P}\left(\bigcup_{i=1}^{100}A_i\right)\\
&= 1-\frac{1}{100^{100}}\sum_{k=1}^{100}(-1)^{k-1}\binom{100}{k}(100-k)^{100-k}\\
&= \frac{1}{100^{100}}\sum_{k=0}^{100}\binom{100}{k}(-k)^{k}.
\end{aligned}$$

   The third equality is by the *inclusion-exclusion formula.*

(b) What is the *expected* number of people who came for the wedding? [5]

This is given by

$$\mathbf{E}[X] = \sum_{i=1}^{100} \mathbf{P}(A_i) = 100 \times \frac{99^{99}}{100^{100}} = \left(\frac{99}{100}\right)^{99} \approx 0.3697.$$

5. There was a strange news reported in the seven o'clock news, apparently a newly built bridge in northern England collapsed during the peak commute hours without any problem reported about it earlier. The total number of casualties was yet to be determined but it was somewhere in between 60 and 80.

Harry was clever to figure out that this must have been a work of the Dark Lord and his followers. He quickly reported it to Professor Dumbledore, who told him that from his prior experiences, he knew that the Dark Lord must had decided on the exact number of people he wanted to harm by either rolling a fair 100 sided die and then taking the number which came on top, or tossing a coin 100 times independently and then took the number of heads. He also did not think that the Dark Lord had any preference on the choice of his method.

When Harry told this to his friend Hermione, she was quick to make a guess about the actual method the Dark Lord used.

(a) What was Hermione's guess? Explain your answer. [6]

Let Z be the exact number of casualties determined by the Dark Lord and let I is the indicator random variable taking value $I = 1$ if he rolled the 100 sided die to determine Z, otherwise it takes the value $I = 0$. Then the conditional distributions of $Z \,\Big|\, I = 1 \sim \text{Uniform}(\{1, 2, \cdots, 100\})$ and $Z \,\Big|\, I = 0 \sim \text{Binomial}\left(100, \frac{1}{2}\right)$. Also $\mathbf{P}(I = 1) = \mathbf{P}(I = 0) = \frac{1}{2}$.

Now the event $A := [60 \leq Z \leq 80]$ has been observed. So

$$\begin{aligned}
\mathbf{P}\left(I = 1 \,\Big|\, A\right) &= \frac{\mathbf{P}\left(A \,\Big|\, I = 1\right)\mathbf{P}(I = 1)}{\mathbf{P}(A)} \\
&= \frac{\mathbf{P}\left(A \,\Big|\, I = 1\right)\mathbf{P}(I = 1)}{\mathbf{P}\left(A \,\Big|\, I = 1\right)\mathbf{P}(I = 1) + \mathbf{P}\left(A \,\Big|\, I = 0\right)\mathbf{P}(I = 0)} \\
&= \frac{\mathbf{P}(60 \leq U \leq 80)}{\mathbf{P}(60 \leq U \leq 80) + \mathbf{P}(60 \leq V \leq 80)}
\end{aligned}$$

where $U \sim \text{Uniform}(\{1, 2, \cdots, 100\})$ and $V \sim \text{Binomial}\left(100, \frac{1}{2}\right)$. So $\mathbf{P}(60 \leq U \leq 80) = \frac{21}{100}$ and $\mathbf{P}(60 \leq V \leq 80) \approx \Phi(6.1) - \Phi(1.9) \approx 1 - \Phi(1.9) = \Phi(-1.9) \approx 0.0287$. Note $\mathbf{E}[V] = 50$ and $\mathbf{Var}(V) = 25$. Thus

$$\mathbf{P}\left(I = 1 \,\Big|\, A\right) \approx \frac{21}{23.87} \approx 0.8798.$$

So Hermione's must had guessed that the Dark Lord rolled the 100 sided die.

(b) Will she make a different guess if the news channel said the number of casualties were between 40 to 60? Explain your answer. [6]

Let $B := [40 \le Z \le 60]$, then similar argument will show

$$\begin{aligned}
\mathbf{P}\left(I = 1 \,\middle|\, B\right) &= \frac{\mathbf{P}\left(B \,\middle|\, I = 1\right)\mathbf{P}\left(I = 1\right)}{\mathbf{P}\left(B\right)} \\
&= \frac{\mathbf{P}\left(B \,\middle|\, I = 1\right)\mathbf{P}\left(I = 1\right)}{\mathbf{P}\left(B \,\middle|\, I = 1\right)\mathbf{P}\left(I = 1\right) + \mathbf{P}\left(B \,\middle|\, I = 0\right)\mathbf{P}\left(I = 0\right)} \\
&= \frac{\mathbf{P}\left(40 \le U \le 60\right)}{\mathbf{P}\left(40 \le U \le 60\right) + \mathbf{P}\left(40 \le V \le 60\right)}
\end{aligned}$$

where $U \sim \text{Uniform}\left(\{1, 2, \cdots, 100\}\right)$ and $V \sim \text{Binomial}\left(100, \frac{1}{2}\right)$. Now $\mathbf{P}\left(40 \le U \le 60\right) = \frac{21}{100}$ and $\mathbf{P}\left(40 \le V \le 60\right) \approx \Phi(2.1) - \Phi(-2.1) = 2\Phi(2.1) - 1 \approx 2 \times 0.9821 - 1 \approx 0.9642$. Once again note that $\mathbf{E}\left[V\right] = 50$ and $\mathbf{Var}\left(V\right) = 25$. Finally,

$$\mathbf{P}\left(I = 1 \,\middle|\, B\right) \approx \frac{21}{117.42} \approx 0.1788.$$

So yes, in that case she must had guessed that the Dark Lord tossed the fair coin 100 times.