Final Exam Solutions, Math 54.2, 18 December 2013

Question 1. 1f, 2c, 3e, 4f, 5c, 6e, 7e, 8b, 9c, 10d, 11b, 12c

Question 2. Auxiliary equation $x^2 + 0.2x + 1.01$, solutions $\lambda_\pm = -0.1 \pm i$.
A basis of homogeneous solutions is $e^{-0.1t}\cos(t)$, $e^{-0.1t}\sin(t)$.
Find a particular solution by the method of undetermined coefficients. Since $\pm i$ is not a root of the equation, we look for $a\cos(t) + b\sin(t)$; substituting in the equation gives

$$(-a + 0.2b + 1.01a)\cos(t) + (-b - 0.2a + 1.01b)\sin(t) \equiv \sin(t)$$

and equating coefficients gives the system

$$\begin{aligned} 0.01a + 0.2b &= 0 \qquad (1)\\ -0.2a + 0.01b &= 1 \qquad (2)\end{aligned}$$

which is solved by $a = -\frac{2000}{401}, b = \frac{100}{401}$.
The general solution is $(c_1e^{-0.1t} + a)\cos(t) + (c_2e^{-0.1t} + b)\sin(t)$. Vanishing at $t = 0$ requires

$$c_1 = -a = \frac{2000}{401}.$$

The derivative is

$$-(c_1e^{-0.1t} + a)\sin(t) - 0.1c_1e^{-0.1t}\cos(t) + (c_2e^{-0.1t} + b)\cos(t) - 0.1c_2e^{-0.1t}\sin(t)$$

and its vanishing at $t = 0$ requires $-0.1c_1 + c_2 + b = 0$, or

$$c_2 = -b + 0.1c_1 = -\frac{100}{401} + \frac{200}{401} = \frac{100}{401}.$$

So the solution is

$$\frac{2000}{401}\left(e^{-0.1t} - 1\right)\cos(t) + \frac{100}{401}\left(e^{-0.1t} + 1\right)\sin(t).$$

(b) Clearly, as t gets large the solution approaches $-\frac{2000}{401}\cos(t) + \frac{100}{401}\sin(t)$ whose amplitude is $\sqrt{2000^2 + 100^2}/401 = \frac{100}{\sqrt{401}}$, around 5. This does not depend on initial conditions, because a change of initial conditions would just change c_1, c_2 and not the a, b found above, and $c_{1,2}$ are coupled to exponentially damped terms. Finally, replacing $\sin(t)$ by $\cos(t)$ in the equation can be implemented by a forward time shift by $\pi/2$, and also does not change the limiting amplitude. (If you go through the system, you will find the limiting solution $b\cos(t) - a\sin(t)$ with the a, b above.)

Question 3. We recognize a candidate for the Fourier cosine series of the function $sin(x)$ on $[0, \pi]$, extended to a π-periodic function on $\mathbf{R}$. Equivalently, we recognize in $|\sin(x)|$ an *even* 2π-periodic function, whose Fourier series only involves cosines. As you know, both of these pictures should lead to the same Fourier series.
Let us check this by computing the Fourier cosine coefficients of $\sin(x)$ on $[0, \pi]$ from the inversion formulae.

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin(x)dx = \frac{1}{\pi}\left(-\cos(x)\right)|_0^\pi = \frac{2}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)dx = \frac{1}{\pi}\int_0^\pi \{\sin[(n+1)x] - \sin[(n-1)x]\}\,dx$$

For $n = 1$, this is just $\frac{1}{\pi}\int_0^\pi \sin(2x)dx = \frac{1}{2\pi}\left(-\cos(x)\right)|_0^\pi = 0$. For $n > 1$, we get

$$\frac{1}{\pi}\left\{-\frac{\cos[(n+1)x]}{n+1} + \frac{\cos[(n-1)x]}{n-1}\right\}|_0^\pi = \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1} - \frac{1-(-1)^{n-1}}{n-1}\right) =$$

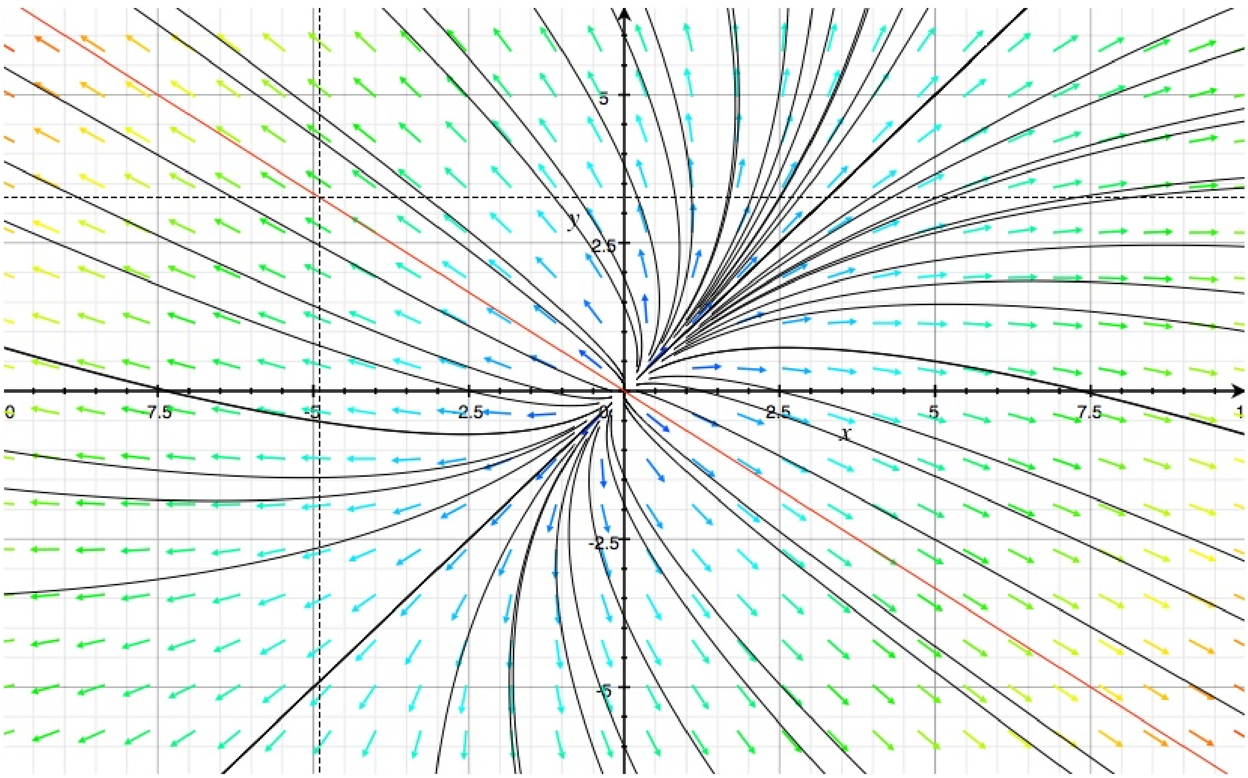
$$= \frac{1-(-1)^{n+1}}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = ((-1)^{n+1} - 1)\frac{2}{\pi}\frac{1}{n^2-1}.$$

This gives $-4/\pi(n^2-1)$ when $n > 0$ is even, and 0 when n is odd, as we needed to find.
Finally, the series converges to $|\sin(x)|$ for every value of x because the function is continuous and piecewise differentiable.

Question 4. Char polynomial $\lambda^2 - 3\lambda + 2$, eigenvalues $\lambda 1, 2 = 1, 2$, eigenvectors $\mathbf{v}_1 = [1, 1]^T$ and $\mathbf{v}_2 = [3, -2]^T$.
The general solution has the form

$$c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + 3c_2 e^{2t} \\ c_1 e^t - 2c_2 e^{2t} \end{bmatrix}$$

for arbitrary constants c_1, c_2.



Eigen-lines in red and black.

EXTRA CREDIT Question The boundary conditions (Neumann) require us to use the Fourier cosine series expansion of $u(x, t)$:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos(nx).$$

Because of the initial condition $u(x, 0) = \sin(x)$, the $u_n(0)$ must be the coefficients of the Fourier cosine series expansion on $[0, \pi]$

$$\sum_{n=0}^{\infty} u_n(0) \cos(nx) = \sin(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{n>0 \\ n \text{ even}}} \frac{\cos(nx)}{n^2 - 1}$$

so that $u_0(0) = \frac{2}{\pi}$, $u_n(0) = 0$ for n odd and $u_n(0) = -\frac{4}{\pi(n^2-1)}$ for $n > 0$ even.
Substituting the series expansion for u in the equation $\partial u/\partial t = \partial^2 u/\partial x^2 + e^{-t} \cos(x)$ we get

$$\sum_{n=0}^{\infty} u_n'(t) \cos(nx) = \sum_{n=0}^{\infty} (-n^2) u_n(t) \cos(nx) + e^{-t} \cos(x)$$

and by equating the coefficients of $\cos(nx)$ on the left and right for each n we get

$$u_n'(t) = -n^2 u_n(t) \text{ for } n \neq 1, \qquad u_1'(t) = -u_1(t) - e^{-t}.$$

So,

$$u_n(t) = \begin{cases} \frac{2}{\pi}, & n = 0 \\ te^{-t}, & \text{for } n = 1 \\ -\frac{4}{\pi(n^2-1)} e^{-tn^2}, & \text{for } n > 1, \text{ even} \\ 0, & \text{for } n > 1, \text{ odd} \end{cases}$$

Note that te^{-t} is the particular solution of the inhomogeneous ODE $y' = -y + e^{-t}$ which vanishes at $t = 0$.