Mathematics 54.2
Final Exam, 18 December 2013
180 minutes, 90 points

NAME: ______________________ **ID:** ______________________

GSI: ____

INSTRUCTIONS:
You must justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.

This is a CLOSED BOOK examination, NO NOTES and NO CALCULATORS are allowed.
NO CELL PHONE or EARPHONE use is permitted.
Please turn in your finished examination to your GSI before leaving the room.

There is an **Extra Credit** question. It will not be added to the score, but a substantially complete answer will bump your **final course grade** by one step (for instance, B+ to A–). Since no partial credit is available for that, you are advised to address the other questions first.

Q1	
Q2	
Q3	
Q4	
Tot	
Letr	
Xtra	

Question 1. (30 points) Choose the correct answers, worth 2.5 points each. No justification needed. No credit for incorrect answers, or if more than one answer is circled.
Please transfer your choices into the table on p.4 or you may miss the credit.

1. Under which of the circumstances below can we be certain that a system $A\mathbf{x} = \mathbf{b}$, with a 5×4 matrix A, has at least one solution?

   (a) Always
   (b) Never
   (c) When A has four pivots
   (d) When A has a left nullspace
   (e) When $\mathbf{b}$ is in $\mathrm{Nul}(A)$
   (f) When $\mathbf{b}$ is in $\mathrm{Col}(A)$

2. The dimensions of the column space and of the nullspace of a matrix add up to

   (a) The number of rows
   (b) The number of pivots
   (c) The number of columns
   (d) The number of solutions of the system
   (e) The number of free variables
   (f) It depends on the pivot positions

3. A number λ is an *eigenvalue* of an $n \times n$ matrix A if and only if:

   (a) $\mathrm{Nul}(A - \lambda I_n) = \{\mathbf{0}\}$
   (b) $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \in \mathbf{R}^n$
   (c) λ is a pivot of A
   (d) $\det(\lambda A) = 0$
   (e) $\det(A - \lambda I_n) = 0$
   (f) $A - \lambda I_n$ is non-singular

4. The following is an eigenvalue of $A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$:

   (a) 2
   (b) 3
   (c) 4
   (d) 5
   (e) 6
   (f) 7

5. In a new basis of $\mathbf{R}^2$, the coordinate vector of the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and that of the vector $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then, the coordinate vector of $\begin{bmatrix} 6 \\ 7 \end{bmatrix}$ is

   (a) $\begin{bmatrix} 8 \\ 7 \end{bmatrix}$
   (b) $\begin{bmatrix} 8 \\ 8 \end{bmatrix}$
   (c) $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$
   (d) $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$
   (e) $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$
   (f) Not determined by the data

6. Under which circumstances is the square matrix A guaranteed to have non-zero determinant?

   (a) A has positive entries on the diagonal
   (b) No entries of A are zero
   (c) A is the coefficient matrix of a consistent linear system
   (d) A has orthogonal columns
   (e) The system $A\mathbf{x} = \mathbf{0}$ has a unique solution
   (f) None of the above

7. A subspace S of $\mathbf{R}^n$ is the *orthogonal complement* of a subspace T if and only if

(a) Some vector in S is orthogonal to some vector in T

(b) Every vector in $\mathbf{R}^n$ has an orthogonal projection to S and to T

(c) Every vector in S is orthogonal to every vector in T

(d) All vectors in S are orthogonal to each other, as are all vectors in T

(e) S contains precisely the vectors in $\mathbf{R}^n$ orthogonal to every vector in T

(f) S contains all the orthogonal vectors of $\mathbf{R}^n$ which are not in T

8. A least-squares solution $\hat{\mathbf{x}}$ of a linear system $A\mathbf{x} = \mathbf{b}$, with A of size $m \times n$, is always characterized by the following:

(a) $\hat{\mathbf{x}}$ is the shortest vector in $\mathrm{Col}(A)$

(b) $\hat{\mathbf{x}}$ is in $\mathbf{R}^n$ and $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ is as short as possible

(c) $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$

(d) $\mathbf{b}$ is the orthogonal projection of $A\hat{\mathbf{x}}$ onto $\mathrm{Col}(A)$

(e) $\hat{\mathbf{x}}$ is in $\mathrm{Col}(A)$ and $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ is as short as possible

(f) $\hat{\mathbf{x}}$ is in $\mathbf{R}^m$ and $\|\hat{\mathbf{x}} - \mathbf{b}\|$ is as short as possible

9. The least-squares solution to $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is

(a) $\mathbf{x} = 0$

(b) $\mathbf{x} = 1$

(c) $\mathbf{x} = 2$

(d) $\mathbf{x} = 3$

(e) $\mathbf{x} = 4$

(f) It does not exist

10. A matrix A is *diagonalizable* if and only if

(a) We can find matrices S, D with $AS = SD$ and D diagonal

(b) We can find matrices S, D with $AS = SD$, D diagonal and A invertible

(c) We can find matrices S, D with $AS = SD$, S diagonal and D invertible

(d) We can find matrices S, D with $AS = SD$, D diagonal and S invertible

(e) We can find matrices S, D with $AS = SD$, D diagonal and S, D both invertible

(f) A has eigenvectors

11. Pick the matrix on the list which is NOT diagonalizable over $\mathbf{C}$, if any; else, pick option (f).

(a) $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(f) All of them are diagonalizable over $\mathbf{C}$

12. The *characteristic polynomial* of a 2×2 matrix A is (recall that $\mathrm{Tr}(A) = a_{11} + a_{22}$)

(a) $\det(\lambda I_2 + A)$

(b) $\lambda^2 - \lambda \det(A) + \mathrm{Tr}(A)$

(c) $\lambda^2 - \lambda \mathrm{Tr}(A) + \det(A)$

(d) $\det(\lambda I_2) - \det(A)$

(e) $\lambda^2 + \lambda \mathrm{Tr}(A) - \det(A)$

(f) $\lambda^2 + \lambda \det(A) + \mathrm{Tr}(A)$

Q1	a	b	c	d	e	f
Q2	a	b	c	d	e	f
Q3	a	b	c	d	e	f
Q4	a	b	c	d	e	f
Q5	a	b	c	d	e	f
Q6	a	b	c	d	e	f

Q7	a	b	c	d	e	f
Q8	a	b	c	d	e	f
Q9	a	b	c	d	e	f
Q10	a	b	c	d	e	f
Q11	a	b	c	d	e	f
Q12	a	b	c	d	e	f

Question 2. (20 points, 15+5)
(a) Solve the following second order ODE with initial conditions $y(0) = y'(0) = 0$:

$$y''(t) + 0.2y'(t) + 1.01y(t) = \sin(t)$$

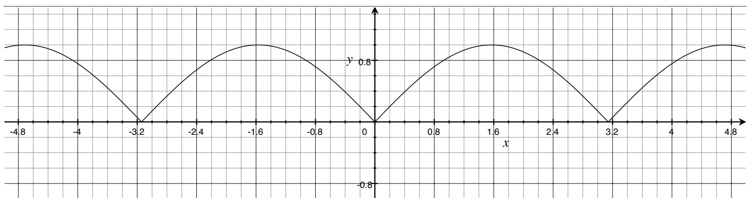
(b) What is the amplitude of the oscillations we get in the limit as t becomes large?
Does that depend on the initial conditions? Would it change with $\cos(t)$ instead of $\sin(t)$ on the right?

Remarks. 1. Recall that the amplitude of a function of the form $a\cos(t) + b\sin(t)$ is $A = \sqrt{a^2 + b^2}$.
(The function can be rewritten in the form $A \cdot \sin(t + \varphi)$, for a well-chosen φ.)
2. You might be able to answer the last parts of (b) without calculations.

Question 3. (20 points)
By using your knowledge of Fourier series (or by any method you choose), prove the identity

$$|\sin(x)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{k>0,\\ k \text{ even}}} \frac{\cos(kx)}{k^2-1}, \quad \text{for all } x \in \mathbf{R}.$$

(The sum runs over the *even* natural numbers.) Clearly state any general facts that you use.
Remark: The following formula may be useful, $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.



The graph of $y = |\sin(x)|$

Question 4. (20 points, 12+8)

(a) Find and write down the general solution of the vector-valued ODE

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1.6 & -0.6 \\ -0.4 & 1.4 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

(b) Draw a 'phase portrait' of this ODE, sketching a few trajectories and velocity vectors, and clearly indicating the way in which the curves cross the coordinate axes.

EXTRA CREDIT Question

There is no point value, but a substantially complete answer will add one step to your final course grade. There is no partial credit. Explain your work; this way, small calculation mistakes will not void your answer.

Solve the inhomogeneous heat equation for a function $u(x,t)$, with Neumann boundary conditions and initial conditions $u(x,0) = \sin(x)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\cos(x); \quad \text{with } \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \quad u(x,0) = \sin(x)$$

Hint: You should find the identity of Question 3 useful.

THIS PAGE IS FOR ROUGH WORK (not graded)