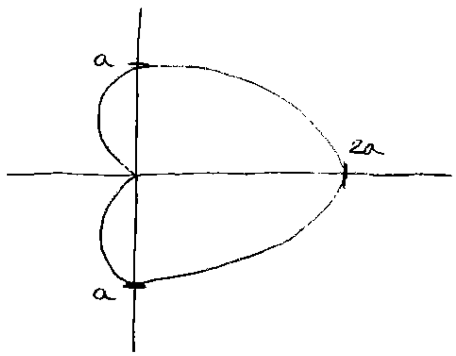
1. (a)

$$r = a(1+\cos\theta) = a + a\cos\theta$$



(b) $$\underbrace{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)} + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\underbrace{\frac{d}{d\theta}\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)}$$

$$\frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{1}{r} = \frac{-\mu r^2}{\ell^2}F(r)$$

$$\frac{2}{r^3}a^2\sin^2\theta + \frac{1}{r^2}a\cos\theta + \frac{1}{r} =$$

$$\frac{2}{r^3}\left(a^2 - a^2\cos^2\theta\right) + \frac{1}{r^2}(r-a) + \frac{1}{r} =$$

$$\frac{2a^2}{r^3} - \frac{2}{r^3}(r-a)^2 + \frac{2}{r} - \frac{a}{r^2} =$$

$$\frac{2a^2}{r^3} - \frac{2}{r} + \frac{4a}{r^2} - \frac{2a^2}{r^3} + \frac{2}{r} - \frac{a}{r^2} =$$

$$\frac{3a}{r^2} = \frac{-\mu r^2}{\ell^2}F(r)$$

$$F(r) = \frac{-3a\ell^2}{\mu r^4}$$

(c) $U = -\int F\,dr = -\frac{a\ell^2}{\mu r^3}$

$T = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2}$

$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = -a\sin\theta\,\dot{\theta} = -a\sin\theta\left(\frac{\ell}{\mu r^2}\right)$

$\longrightarrow T = \frac{\mu}{2}a^2\sin^2\theta\,\frac{\ell^2}{\mu^2 r^4} + \frac{1}{2}\frac{\ell^2}{\mu r^2}$

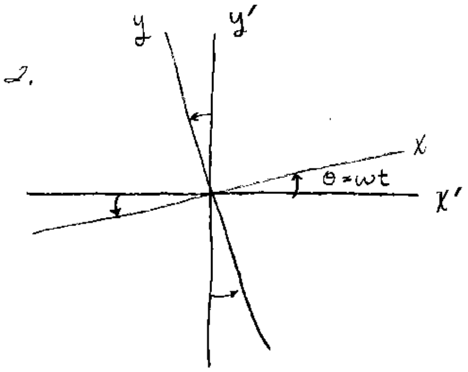
$a^2\sin^2\theta = a^2 - a^2\cos^2\theta = a^2 - (r-a)^2 = 2ar - r^2$

$T = \left(ar - \frac{r^2}{2}\right)\frac{\ell^2}{\mu r^4} + \frac{1}{2}\frac{\ell^2}{\mu r^2} = \frac{a\ell^2}{\mu r^3}$

$E = T + U = 0$

(d) since $E=0$ before & after, no work is required

2.



a) $L = T - V = T = \frac{m}{2}\left(\dot{x}'^2 + \dot{y}'^2\right)$

b) $x = x'\cos\theta + y'\sin\theta$ OR $x' = x\cos\theta - y\sin\theta$

$y = y'\cos\theta - x'\sin\theta$ $\quad y' = y\cos\theta + x\sin\theta \quad (\theta = \omega t)$

$$\dot{x}' = \dot{x}\cos\theta - \omega x\sin\theta - \dot{y}\sin\theta - \omega y\cos\theta$$

$$\dot{y}' = \dot{y}\cos\theta - \omega y\sin\theta + \dot{x}\sin\theta + \omega x\cos\theta$$

$$L = T = \frac{m}{2}\left(\dot{x}'^2 + \dot{y}'^2\right) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \omega^2 x^2 + \omega^2 y^2 - 2\omega\dot{x}y + 2\omega x\dot{y}\right)$$

c) $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - m\omega y$

$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + m\omega x$

$$H = \dot{x}p_x + \dot{y}p_y - L = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 - \omega^2(x^2 + y^2)\right]$$

$$= \frac{m}{2}\left[\left(\frac{p_x}{m} + \omega y\right)^2 + \left(\frac{p_y}{m} - \omega x\right)^2 - \omega^2\left(x^2 + y^2\right)\right]$$

$$= \frac{p_x^2 + p_y^2}{2m} + \omega\left(p_x y - p_y x\right)$$

d) $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} + \omega y \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} - \omega x$

$\dot{p}_x = \frac{-\partial H}{\partial x} = \omega p_y \qquad \dot{p}_y = \frac{-\partial H}{\partial y} = -\omega p_x$

e) $m\ddot{\vec{r}} = -2m(\vec{\omega}\times\vec{v}_r) - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

where $\vec{r} = (x, y, 0)$

$\vec{v}_r = (\dot{x}, \dot{y}, 0)$

$\vec{\omega} = (0, 0, \omega)$

$\rightarrow (\ddot{x}, \ddot{y}, 0) = (2\omega\dot{y}, -2\omega\dot{x}, 0) + (\omega^2 x, \omega^2 y, 0)$

by computing cross products

thus $\ddot{x} = \omega^2 x + 2\omega\dot{y}$

$\ddot{y} = \omega^2 y - 2\omega\dot{x}$

f) From (d), $\dot{x} = \frac{p_x}{m} + \omega y$ (E of M for $\dot{x}$)

$\ddot{x} = \dot{p}_x/m + \omega\dot{y}$ ($\frac{d}{dt}$)

$= \omega p_y/m + \omega\dot{y}$ (E of M for p_x)

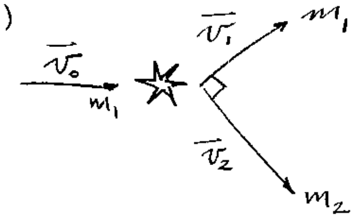
$= \frac{\omega}{m}(m\dot{y} + m\omega x) + \omega\dot{y}$ (def'n of p_y)

$= \omega^2 x + 2\omega\dot{y}$ (SAME AS IN PT. E)

The same calculation holds for $\ddot{y}$, so the results agree

3.

(a)



cons. of $\vec{p}$: $m_1 \vec{v}_0 = m_1 \vec{v}_1 + m_2 \vec{v}_2$

square both sides : $m_1^2 v_0^2 = m_1^2 v_1^2 + m_2^2 v_2^2 \quad (\vec{v}_1 \cdot \vec{v}_2 = 0)$

divide by $2m_1$: $\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \left(\frac{m_2}{m_1}\right) v_2^2$

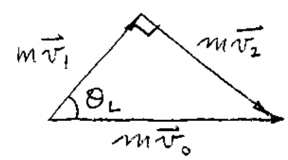
$\updownarrow$ compare

cons. of E : $\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

both expressions are consistent only if $\left(\frac{m_2}{m_1}\right) = 1$ or $\boxed{m_1 = m_2}$

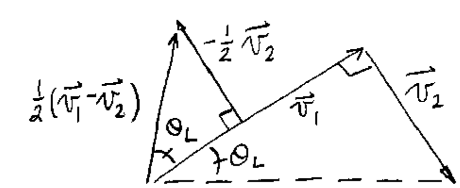
(b)

In lab frame,



In com frame subtract off $\vec{v}_{cm} = \frac{\vec{v}_0}{2} = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)$

com frame: $\vec{v}_1^{\,cm} = \vec{v}_1 - \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2 = \frac{1}{2}(\vec{v}_1 - \vec{v}_2)$



these are similar Δs

Thus $\boxed{\theta_c = 2\theta_L}$

(c) Here $\tan\left(\frac{\theta}{2}\right) = \frac{k}{mbv_0^2} \implies \sec^2\frac{\theta}{2}\left|d\theta\right| = \frac{2k}{mb^2v_0^2}\left|db\right|$

$$\left|\frac{db}{d\theta}\right| = \left(\frac{mbv_0^2}{k}\right)^2 \cdot \frac{k}{2mv_0^2} \cdot \sec^2\frac{\theta}{2} = \cot^2\frac{\theta}{2} \cdot \sec^2\frac{\theta}{2} \cdot \frac{k}{2mv_0^2} = \csc^2\frac{\theta}{2} \cdot \frac{k}{2mv_0^2}$$

$$\sigma = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{k}{mv_0^2}\cot\frac{\theta}{2}\csc\theta\,\csc^2\frac{\theta}{2}\,\frac{k}{2mv_0^2} = \frac{k^2}{4m^2v_0^4} \cdot \frac{1}{\sin^4\frac{\theta}{2}}$$