Directions: This is a *closed* book exam. No calculators, cell phones, pagers, mp3 players and other electronic devices are allowed.
Remember: Answers without explanations will not count. You should **show your work**. Solve each problem on its own page. If you need extra space you can use backs of the pages and the extra page attached to your exam paper, but make a note you did so.

Problem	Score
1	20
2	20
3	18
4	10
5	17
6	20
Total	115
Grade	A+

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

Find *rank* A, a basis for *Col* A and a basis for *Row* A.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\cancel{\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}}$$

$$\cancel{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

A basis for Col A = $\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}}$

basis for row A = $\boxed{\left\{ \begin{matrix} (1, 1, 1) \\ (0, 1, 2) \end{matrix} \right\}}$

$\boxed{\text{rank } A = 2 \ (2 \text{ pivots})}$

$$1\begin{vmatrix}2&3\\1&3\end{vmatrix}+1\begin{vmatrix}1&2\\2&1\end{vmatrix}$$

$$3+-3=0$$

(20) 2. **Problem 2.** Compute (or if undefined say so, explaining why)

5 a) A^{-1}, $A=\begin{bmatrix}1&2&3\\2&1&3\\1&0&1\end{bmatrix}$

undefined since $\det(A)=0$ so not invertible.

$$\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&1&3&0&1&0\\1&0&1&0&0&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-3&-3&-2&1&0\\0&-2&-2&-1&0&1\end{array}\right]$$

5 b) ABA, $A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, $B=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$

$$AB=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}4&5&6\\7&8&9\\1&2&3\end{bmatrix}$$

$$(AB)A=\begin{bmatrix}4&5&6\\7&8&9\\1&2&3\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}6&4&5\\9&7&8\\3&1&2\end{bmatrix}$$

5 c) $\begin{bmatrix}1&2\\2&4\\4&8\end{bmatrix}^{2}[1\ 2\ 4]^{3}$

Undefined a 3×2 cannot multiply a 1×3 since the # of columns of the first matrix don't match up to # of rows of second.

5 d) $\det\begin{bmatrix}3&0&0&5&0\\9&1&7&5&0\\1&4&7&5&2\\1&0&0&3&0\\2&1&0&6&0\end{bmatrix}=2\begin{vmatrix}3&0&0&5\\9&1&7&5\\1&0&0&3\\2&1&0&6\end{vmatrix}=2\cdot(-7)\begin{vmatrix}3&0&5\\1&0&3\\2&0&6\end{vmatrix}$

$$=2(-7)\cdot(-1)\begin{vmatrix}3&5\\1&3\end{vmatrix}=14\cdot4=56$$

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(20) 3. a) State Cramer's rule.

For some system of linear equations or for a matrix representation $Ax = b$

$$x_i = \frac{\det A_i(j)}{\det A}$$

-2 A invertible

b) Use it to solve the linear system (no credit for solving the system directly)

$1 \quad \frac{1}{2} \quad \frac{3}{2}$

$$\begin{cases} x_1 + 2x_2 = 7 \\ x_2 + 3x_3 = 5 \\ x_1 - 2x_3 = 3. \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\det A = 1\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= -2 + 6 = 4$$

$$A_{1j} = \begin{bmatrix} 7 & 2 & 0 \\ 5 & 1 & 3 \\ 3 & 0 & -2 \end{bmatrix}$$

$$\det A_{1j} = 7\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} - 2\begin{vmatrix} 5 & 3 \\ 3 & -2 \end{vmatrix} = -14 + 38 = 24$$

$$A_{2j} = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 5 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

$$\det A_{2j} = 1\begin{vmatrix} 5 & 3 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 7 & 0 \\ 5 & 3 \end{vmatrix} = -19 + 21 = 2$$

$$A_{3j} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\det A_{3j} = 1\begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 7 \\ 1 & 5 \end{vmatrix} = 3 + 3 = 6$$

$$\boxed{x_1 = \frac{24}{4} = 6 \qquad x_2 = \frac{2}{4} = \frac{1}{2} \qquad x_3 = \frac{6}{4} = \frac{3}{2}}$$

(20) 4. Mark each statement True or False. Justify your answers.

a) If $AB = 0$ for two square matrices A, B, then either $A = 0$, or $B = 0$.

False, for example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\text{Both not } 0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) The set $P_2[X, Y]$ of all polynomials in X and Y of degree at most 2 (together with the usual addition and multiplication by a constant) is a vector space of dimension 6.

P_2 = polynomials of the form $a_1x^2 + a_2x + a_3x^0$, $b_1y^2 + b_2y + b_3y^0$ $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

True, this is a vector space of dimension 6 since a single P_2 (all polynomials in x) is isomorphic to R^3. The same is true for Y and y is a seperate polynomial set apart from x. Thus, the set $P_2[X, Y]$ is isomorphic to R^6

$$\begin{array}{cccccc} x^2 & x & x^0 & y^2 & y & y^0 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis that consists of 6 vectors. Note the constants a_3 and b_3 (x^0 and y^0) do not overlap since they are constants in X and Y respectively.

(20) 5. Let P_4 denote the vector space of polynomials of degree at most 4 (vector space together with the adition and multiplication by a constant). Consider the differentiation map $D : P^4 \to P^4$ given by $Df = f'$.

8 a) Show that D is linear.

$u, v \in P_4$ $u = a_1 t^4 + a_2 t^3 + a_3 t^2 + a_4 t + a_5$ $v = b_1 t^4 + b_2 t^3 + b_3 t^2 + b_4 t + b_5$

~~1) $D(u)$ is a subset of P_4 since the differentiation of polynomials of degree at most~~ 3.

✓ 2) closed under addition $D(u+v) = \frac{d}{dt}\left((a_1+b_1)t^4 + (a_2+b_2)t^3 + (a_3+b_3)t^2 + (a_4+b_4)t + (a_5+b_5)\right)$

$= 4(a_1+b_1)t^3 + 3(a_2+b_2)t^2 + 2(a_3+b_3)t + (a_4+b_4) = D(u) + D(v)$

✓ 3) closed under multiplication $D(cu) = \frac{d}{dt}\left(ca_1 t^4 + ca_2 t^3 + ca_3 t^2 + ca_4 t + ca_5\right)$

$= 4ca_1 t^3 + 3ca_2 t^2 + 2ca_3 t + ca_4 = c\,D(u)$

✓ Extra check: 0 element ~~exists~~ $D(0) = 0$

4 b) Find a basis in P_4.

a basis would be $\boxed{\{t^4, t^3, t^2, t, 1\}}$

why

symbolic representation for next problem

$$\begin{array}{ccccc} t^4 & t^3 & t^2 & t & 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix for $A = \begin{bmatrix} D(e_1) & D(e_2) & D(e_3) & D(e_4) & D(e_5) \end{bmatrix}$ (transformation matrix)

5 c) Find the matrix of D in your chosen basis.

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$e_1 = t^4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $D\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4t^3$

$e_2 = t^3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $D\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = 3t^2$

$e_3 = t^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $D\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = 2t$

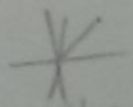
$e_4 = t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $D\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1$

$e_5 = 1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $D\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$

(20) 6. Mark each statement True or False. Justify your answers.

a) If there is a linear transformation $T : \mathbb{R}^5 \to V$ which is onto, then $dim\, V \geq 5$.

False, this is simply an existence question. For every element in V, there is a corresponding element in R^5 that is mapped by T into V. However, suppose V is the "result" of a mapping of R^5 into a line (a "projection"). Thus $\dim V$ is 1 which is < 5

Just as in R^3  Mapping R^3 onto the xy plane is onto since all the points in the plane can be mapped, but $\dim V = 2$

At most $\dim V \leq 5$ since a mapping cannot increase the dimensions from the domain to the range. It is bounded by its 5 pivot columns.

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b) Any linearly independent set in $\mathbb{R}^3$ must have exactly three elements.

False, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set with only 2 elements.

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