Midterm 1 solutions

February 25, 2014

1. When we put the sphere in the box, there is a gap of $L - 2R$ between the side of the box and the surface of the sphere. Assuming that the thermal expansion happens isotropically, the sides of the box and the radius of the sphere change exactly the same whay in every direction. If we change the temperature from T_0 to $T_0 + \Delta T$ then the radius of the sphere and the side of the box change as follows:

   $$R' = R + \Delta R = R + \alpha_{\mathrm{Au}} R \Delta T L' = L + \Delta L = L + \alpha_{\mathrm{plexi}} L \Delta T$$

   So at the new temperature, the gap becomes:

   $$L' - 2R' = L - 2R - (2\alpha_{\mathrm{Au}} R - \alpha_{\mathrm{Plexi}} L)\Delta T$$

   The sphere fits in the box as long as this gap $\geqslant 0$. This corresponds to:

   $$\Delta T \leqslant \frac{L - 2R}{2\alpha_{\mathrm{Au}} R - \alpha_{\mathrm{Plexi}} L}$$

   or

   $$T_m = T_0 + \frac{L - 2R}{2\alpha_{\mathrm{Au}} R - \alpha_{\mathrm{Plexi}} L}$$

2. First let's write down the amounts of heat involved in cooling the water and heating the ice to 0°C:

   $$Q_{\mathrm{water}} = m_{\mathrm{water}} C_{\mathrm{water}} \Delta T_{\mathrm{water}} \qquad Q_{\mathrm{ice,heating}} = m_{\mathrm{ice}} C_{\mathrm{ice}} \Delta T_{\mathrm{ice}}$$

   Since there is no net heat flow to or from outside, all the heat released by the water is used to melt the ice. Assuming that an amount $m_{\mathrm{ice,melts}}$ is melted at the end of this heat transfer, we have the following equation for the heats:

   $$m_{\mathrm{ice,melts}}\, L_{\mathrm{fusion}} + Q_{\mathrm{ice,heating}} + Q_{\mathrm{water}} = 0$$

   Solving for the mass, using $\Delta T_{\mathrm{water}} = -15°\mathrm{C}$ and $\Delta T_{\mathrm{ice}} = 10°\mathrm{C}$:

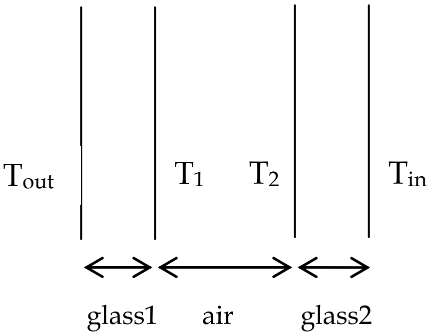
   $$m_{\mathrm{ice,melts}} = -\frac{1}{L_{\mathrm{fusion}}}(m_{\mathrm{water}} C_{\mathrm{water}} \Delta T_{\mathrm{water}} + m_{\mathrm{ice}} C_{\mathrm{ice}} \Delta T_{\mathrm{ice}}) = 0.176 \text{ kg}$$

   Thus the equilibrium state has 24 g of solid ice and 1.176 kg of liquid water. As for the entropy, before the system was in equilibrium, there was heat transfer from a body with higher temperature to a body with lower temperature (liquid water above 0°C to solid ice below and at 0°C). Any such transfer of heat is irreversible, so $\Delta S > 0$ over the course of this transformation.

Problem 3 – Conductive heat transfer (20 pts)

Quantitatively compare the daily heat loss of a house by conduction through a single-pane and a double-pane (see cross-section below) window of same surface area (2 m^2). The temperature is assumed to be 20°C inside and 15°C outside. The glass slabs are 3 mm thick and the air layer of the double-pane window is 4 mm thick.

$k_{glass} = 0.84$ *J/m.s.K*
$k_{air} = 0.023$ *J/m.s.K*



Sol.

Daily heat loss is proportional to rate of heat loss.

For one slab, the rate of heat loss is:

$$\left(\frac{dQ}{dt}\right)_1 = \frac{k_g A}{L_g}(T_{out} - T_{in})$$

Two slabs, the rate of heat loss of each layer is:

$$\left(\frac{dQ}{dt}\right)_{glass1} = \frac{k_g A}{L_g}(T_{out} - T_1)$$

$$\left(\frac{dQ}{dt}\right)_{air} = \frac{k_a A}{L_a}(T_1 - T_2)$$

$$\left(\frac{dQ}{dt}\right)_{glass2} = \frac{k_g A}{L_g}(T_2 - T_{in})$$

When T1 and T2 don't fluctuate with time,

$$\left(\frac{dQ}{dt}\right)_{glass1} = \left(\frac{dQ}{dt}\right)_{air} = \left(\frac{dQ}{dt}\right)_{glass2}$$

Therefore,

$\frac{k_g A}{L_g}(T_{out} - T_1) = \frac{k_a A}{L_a}(T_1 - T_2)$ -----(1)

$\frac{k_a A}{L_a}(T_1 - T_2) = \frac{k_g A}{L_g}(T_2 - T_{in})$ -----(2)

Solve (1),

$T_1 = \frac{aT_{out}+T_2}{1+a}$ where $a = \frac{L_a k_g}{L_g k_a} \cong 49$

Plug into (2), get

$$T_2 = \frac{1}{2+a}T_{out} + \frac{1+a}{2+a}T_{in}$$

Therefore,

$$\left(\frac{dQ}{dt}\right)_2 = \left(\frac{dQ}{dt}\right)_{glass2} = \frac{k_g A}{L_g}\left(\frac{1}{2+a}T_{out} - \frac{1}{2+a}T_{in}\right) = \frac{1}{2+a}\left(\frac{dQ}{dt}\right)_1 \cong 0.02\left(\frac{dQ}{dt}\right)_1$$

Problem 4 – Ideal gases (20 pts)

1 mole of a polyatomic (non linear) ideal gas undergoes a reversible thermodynamic process from pressure and volume (P_a, V_a) to volume $V_b=2V_a$, following the curve $TV=const$.

a) Draw the corresponding path on a P-V diagram, along with the isothermal and adiabatic processes between the same volumes V_a and V_b for comparison.

b) Calculate the work done by the gas from *a* to *b*, and represent it graphically on the P-V diagram.

c) Calculate the change in internal energy, the heat gained by the gas.

d) Calculate the change in entropy of the gas from *a* to *b*, and determine, without any calculation, the total change in entropy of the closed system {gas + environment}.

Sol.

Polyatomic ideal gas has six degree of freedom, three from translational kinetic energy, three from rotational energy. $\gamma = \frac{4}{3}$

(a)

With ideal gas law,

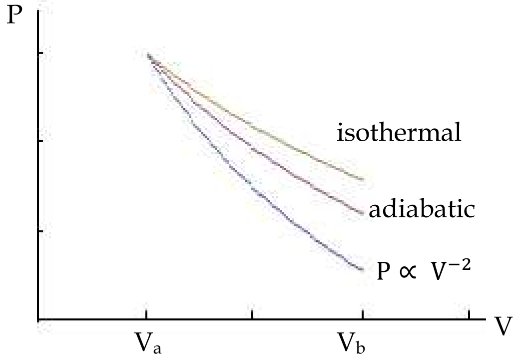
$$PV = RT$$

$$\Rightarrow T = \frac{PV}{R} \;\Rightarrow\; TV = \frac{PV^2}{R} = const. = \frac{P_aV_a{}^2}{R}$$

$$\Rightarrow P \propto V^{-2}$$

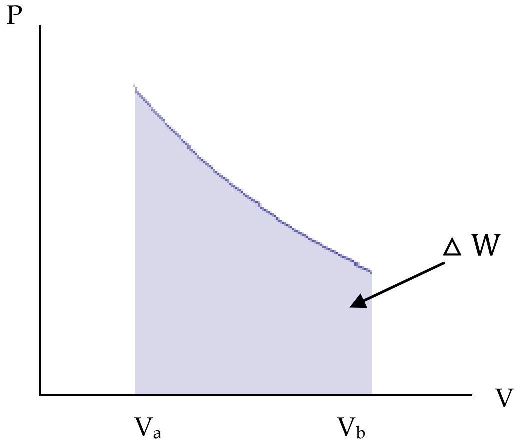
For isothermal process: $P \propto V^{-1}$

For adiabatic process: $P \propto V^{-\gamma} \propto V^{-4/3}$



(b)

$$\Delta W = \int P\,dV = \int_{V_a}^{2V_a} \frac{P_aV_a{}^2}{V^2}\,dV = P_aV_a/2$$



(c)

From ideal gas law,

$$T_a = \frac{P_a V_a}{R}$$

Since TV=const., $T_a V_a = T_b V_b \ \Rightarrow \ T_b = \frac{T_a}{2}$

Internal energy change is

$$\triangle E = 3R \triangle T = 3R(T_b - T_a) = -\frac{3}{2} R T_a = -\frac{3}{2} P_a V_a$$

By first law of thermodynamics,

$$Q = \triangle E + \triangle W = -P_a V_a$$

(d)

The entropy change of gas is

$$\triangle S_g = \int \frac{dQ}{T} = \int \frac{dE}{T} + \int \frac{dW}{T} = \int \frac{3RdT}{T} + \int \frac{PdV}{T} = 3R\ln(\frac{T_b}{T_a}) + \int \frac{R}{V} dV$$

$$\triangle S_g = 3R\ln\left(\frac{T_b}{T_a}\right) + R\ln\left(\frac{V_b}{V_a}\right) = -2R\ln 2$$

Since it is reversible process,

$$\triangle S_{env} = \int \frac{dQ_{env}}{T}$$

The heat flows out from the gas goes to environment, therefore,

$$dQ_{env} = -dQ_g$$

So

$$\triangle S_{env} = - \triangle S_g$$

$$\triangle S = \triangle S_{env} + \triangle S_g = 0$$

Problem 5

There are a few ways to do this problem. You could get points by taking any of these approaches. One method involves the average speed of the distribution, so let us compute that. I get (noting that the integral is right on the equation sheet):

$$\bar{v} = \int_0^\infty v 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}} dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{1}{2(\frac{m}{2kT})^2} = \sqrt{\frac{8kT}{\pi m}}$$

I can compute $\bar{v} = 737$ m/s from the distribution given, so:

$$T = \frac{\pi m \bar{v}^2}{8k} = 517\ K$$

Another way to do this is to look at the most probable speed. I would call this the "I don't have a calculator method." For this, I need to take a derivative of the Maxwell-Boltzmann distribution and set it equal to zero:

$$0 = f'(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv_p^2}{2kT}} \left(2v_p - \frac{mv_p^3}{kT}\right)$$

This will be true if:

$$v_p = \sqrt{\frac{2kT}{m}}$$

So the temperature is (using 600 m/s as the most probable speed):

$$T = \frac{mv_p^2}{2k} = 437\ K$$

You also could have used the temperature-average kinetic energy relationship given on the formula sheet to write:

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

So computing the velocity-temperature relationship was easy here, but the RMS velocity is a little more annoying to calculate from the given numbers. I get that $v_{rms} = 797$ m/s. This gives:

$$T = \frac{mv_{rms}^2}{3k} = 514\ K$$

Since neon is monatomic, it has 3 degrees of freedom. So:

$$E_{int} = \frac{3}{2}NkT = \{1.07 \text{ kJ}, 0.91 \text{ kJ}, 1.06 \text{ kJ}\}$$

Where the three answers are the answers you would get for the three methods above. If the gas were diatomic instead of monatomic, the values of the temperature would not change since nothing in the Maxwell-Boltzmann distribution depends on the number of degrees of freedom. The internal energy will increase by a factor of $\frac{5}{3}$ since diatomic molecules have 5 degrees of freedom.

NOTE: You may be wondering why the three temperatures above are not all equal. This is because we sampled the distribution and binned it in a histogram, which results in some errors in values that we get from it when compared to the ideal distribution (and it turns out v_p picks up a lot more error from this than $\bar{v}$ and v_{rms}). If we knew this distribution to better precision and the distribution is actually Maxwell-Boltzmann, these would be equal (and the $\bar{v}$ and v_{rms} methods are really close).