Practice Midterm 1 , Math 53

1. (a) Compute ∇f for $f(x,y) = e^{x^2y+\sin(xy)}$.

   We have $f_x(x,y) = (2xy + y\cos(xy))e^{x^2y+\sin(xy)}$ and $f_y(x,y) = (x^2 + x\cos(xy))e^{x^2y+\sin(xy)}$, so

$$\nabla f(x,y) = \langle (2xy + y\cos(xy))e^{x^2y+\sin(xy)}, (x^2 + x\cos(xy))e^{x^2y+\sin(xy)} \rangle.$$

   You can also write your answer as $\nabla f(x,y) = e^{x^2y+\sin(xy)}\langle 2xy + y\cos(xy), x^2 + x\cos(xy)\rangle$ or $\nabla f(x,y) = e^{x^2y+\sin(xy)}(2xy + y\cos(xy)\mathbf{i} + x^2 + x\cos(xy)\mathbf{j})$.

   (b) Compute ∇g for $g(x,y,z) = (x^2 + y^3 + z^4)^{-1}$.

   We have $g_x = -2x(x^2 + y^3 + z^4)^{-2}$, $g_y = -3y^2(x^2 + y^3 + z^4)^{-2}$ and $g_z = -4z^3(x^2 + y^3 + z^4)^{-2}$. Then

$$\nabla g(x,y,z) = \langle -2x(x^2 + y^3 + z^4)^{-2}, -3y^2(x^2 + y^3 + z^4)^{-2}, -4z^3(x^2 + y^3 + z^4)^{-2} \rangle.$$

   You can also write your answer as $\nabla g(x,y,z) = (x^2+y^3+z^4)^{-2}\langle -2x, -3y^2, -4z^3\rangle$ or $\nabla g(x,y,z) = (x^2 + y^3 + z^4)^{-2}(-2x\mathbf{i} - 3y^2\mathbf{j} - 4z^3\mathbf{k})$.

2. Find the critical points of the function $f(x,y) = x^4 + 2y^2 - 4xy$, and classify each as a local maximum, local minimum or saddle point.

   The partial derivatives of f are: $f_x(x,y) = 4x^3 - 4y$, $f_y(x,y) = 4y - 4x$. Setting both equal to zero gives the system of equations $y = x^3$ and $y = x$. This is easily solved to obtain

$$\text{Critical points: } (0,0), (1,1), (-1,-1)$$

   To classify them we use the second order test, so we need the second order partials derivatives of f: $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 4$ and $f_{xy}(x,y) = f_{yx}(x,y) = -4$. Then $D(x,y) = 48x^2 - 16$. Evaluating at the critical points we get:
   $D(0,0) = -16 < 0$ thus

$$(0,0) \text{ is a saddle point.}$$

   $D(1,1) = 32 > 0$ and $f_{xx}(1,1) = 12 > 0$, thus

$$(1,1) \text{ is a local minimum.}$$

   Finally, $D(-1,-1) = 32 > 0$ and $f_x x(-1,-1) = 12 > 0$, thus

$$(-1,-1) \text{ is a local minimum.}$$

3. The position vector $\mathbf{r}(t)$ of a particle moving in three dimensions satisfies $\mathbf{r}' = \mathbf{r} \times \mathbf{a}$, where $\mathbf{a}$ is a fixed vector. Show that either the particle is not moving or else its motion lies within a circle. (Hint: Show $|\mathbf{r}|$ and $\mathbf{r} \cdot \mathbf{a}$ are constant).

   We start by proving the hint. We first show that $|\mathbf{r}|^2$ is constant (and hence $|\mathbf{r}|$ is constant):

$$\frac{d|\mathbf{r}|^2}{dt} = \frac{d(\mathbf{r}\cdot\mathbf{r})}{dt} = 2\mathbf{r}\cdot\mathbf{r}' = 2\mathbf{r}\cdot(\mathbf{r}\times\mathbf{a}) = 2(\mathbf{r}\times\mathbf{r})\cdot\mathbf{a} = 0,$$

because $\mathbf{r} \times \mathbf{r} = 0$. In the last line we use that for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we have the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Since the derivative in time of $|\mathbf{r}(t)|^2$ is zero, we conclude that it is constant.

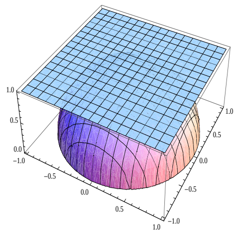
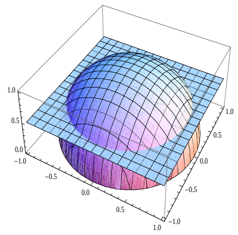
We now show that $\mathbf{r} \cdot \mathbf{a}$ is constant:

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{a}) = \mathbf{r}' \cdot \mathbf{a} + \mathbf{r} \cdot \mathbf{a}' = \mathbf{r}' \cdot \mathbf{a} = (\mathbf{r} \times \mathbf{a}) \cdot \mathbf{a} = \mathbf{r} \cdot (\mathbf{a} \times \mathbf{a}) = 0.$$

We have proved that both $|\mathbf{r}|$ and $\mathbf{r} \times \mathbf{a}$ are constant. We now conclude: We distinguish two cases

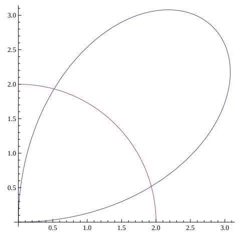
Case 1: $|\mathbf{r}| = 0$. In this case $r(t) = \langle 0, 0, 0 \rangle$ for all time, so the particle is not moving.

Case 2: $|\mathbf{r}| = d > 0$. In this case the particle moves in a sphere of radius d and since it also satisfies $\mathbf{r} \times \mathbf{a} = e$, for some constant e, it also moves in a plane ($\mathbf{r} \times \mathbf{a} = e$ is the equation of a plane with normal vector $\mathbf{a}$). Then the particle moves in the intersection of the sphere and the plane which is either a point, if the surfaces are tangent, and thus the particle is not moving; or the intersection is a circle, and so the motion lies within a circle.

4. Find the area of the region inside the curve $r = 4\sin 2\theta$ and outside the circle $r = 2$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$. (Reminders: $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$.)

The two curves are shown in the picture below. We first find the intersections: $4\sin 2\theta = 2$ where $\theta \in [0, \frac{\pi}{2}]$. This gives us $\sin 2\theta = \frac{1}{2}$ so $\theta = \frac{\pi}{12}$ or $\theta = \frac{5\pi}{12}$.



Then the area of the region is

$$\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (4\sin 2\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2)^2 d\theta = 4\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (1-\cos 4\theta) d\theta - \frac{2\pi}{3} \\
&= \frac{2\pi}{3} - \sin 4\theta \Big|_{\frac{\pi}{12}}^{\frac{5\pi}{12}} = \frac{2\pi}{3} - (\sin(\frac{5\pi}{3}) - \sin(\frac{\pi}{3})) \\
&= \frac{2\pi}{3} + \sqrt{3}.
\end{aligned}$$

5. Assume that the two equations $f(x, y, z) = 0$, $g(x, y, z) = 0$ together implicitly define y as a function of x and z as a function of x. Find formulas for $y' = \frac{dy}{dx}$ and $z' = \frac{dz}{dx}$ in terms of the partial derivatives of f and g.

Using the chain rule we differentiate the two equations $f(x, y(x), z(x)) = 0$ and $g(x, y(x), z(x)) = 0$ with respect to x to obtain:

$$f_x + f_y y' + f_z z' = 0 \text{ and } g_x + g_y y' + g_z z' = 0.$$

This is a linear system for y' and z'. Multiplying the first equation by g_z and the second by f_z and subtracting we obtain, after some algebraic manipulation, that

$$y' = \frac{f_z g_x - g_z f_x}{f_y g_z - g_y f_z}.$$

Similarly we obtain

$$z' = \frac{f_x g_y - g_x f_y}{f_y g_z - g_y f_z}.$$