Midterm 2

1

$\rho(r) = ar$

← non-conducting

$$Q_{enc} = 4\pi \int_0^{r} \underbrace{ar}_{\rho(r)}\, r^2\, dr = \frac{4\pi\, a r^4}{4} = \pi a r^4$$

Using Gauss,

$$\int \vec{E} \cdot d\vec{a} \overset{\text{symmetry}}{=} 4\pi r^2 E_r(r) = k\, Q_{enc}$$

$$\vec{E}_r(r) = \frac{k\, Q_{enc}}{4\pi r^2}\,\hat{r}$$

$$\vec{E}(r) = \begin{cases} \dfrac{k a r^2}{4}\,\hat{r} & r < R \\[2ex] \dfrac{k a R^4}{4 r^2}\,\hat{r} & r > R \end{cases}$$

(a) $\phi(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{s}$

$\phi(\infty) = 0$

r > R:

$$\phi(r) = -\frac{k a R^4}{4} \int_{\infty}^{r} \frac{1}{r^2}\, dr = \frac{k a R^4}{4 r}$$

r < R:

$$\phi(r) = -\int_{\infty}^{R} \frac{k a R^4}{4 r^2}\, dr - \int_{R}^{r} \frac{k a\, r^2}{4}\, dr$$

$$\phi(r) = \frac{k a R^3}{4} - \frac{k a}{12}\left(r^3 - R^3\right)$$

(b) $\vec{E}(r) = \begin{cases} \dfrac{kar^2}{4}\hat{r} & r<R \\ \dfrac{kaR^4}{4r^2}\hat{r} & r>R \end{cases}$ from above.

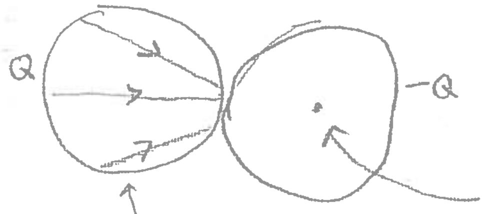
(c) $\vec{\nabla}\cdot\vec{E} = 4\pi k\rho$

Inside sphere, $\vec{\nabla}\cdot\vec{E} \neq 0$, outside sphere $\vec{\nabla}\cdot\vec{E} = 0$.

(d) For conductors, all charge resides on surface. $\vec{E}_{out}$ remains the same (Q_{TOT} same + Gauss), but $\vec{E}_{in}$ now zero.

2 Superposition!

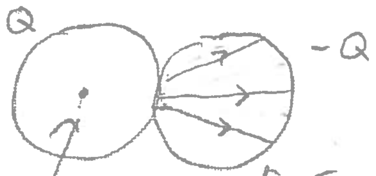
(a)



Outside (-Q) shell, field from (-Q) shell same as (-Q) point charge at center. (Gauss + symmetry)

E inside due to (+Q) shell is zero.

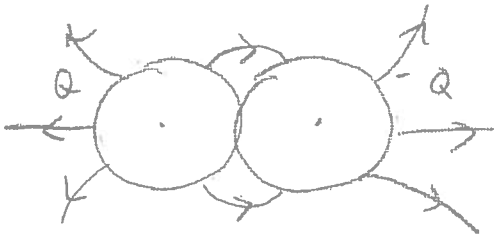
(b)



E inside (-Q) shell due to (+Q) shell is zero.

outside (+Q) shell field is same as (+Q) point charge at center. (Gauss + Symmetry)

(c)

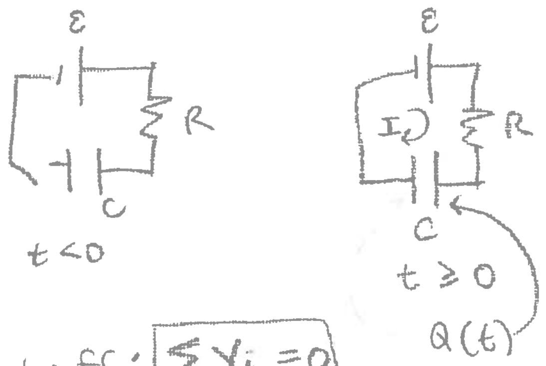


outside both shells,
Field is same as having
a dipole w/ separation $2R$,
again, gauss + symmetry.

(d) Work is same as separating 2 point charges w/ $\pm Q$ & separation $2R$ due to the fact that the average value of ϕ on the shell is is the same as ϕ at the center. (see pg. 87, Purcell).

$$\therefore \quad W_{sep} = -Q\phi = \frac{kQ^2}{(2R)} \quad \left[> 0 \quad \text{because charges are attracted to each other.} \right]$$

3.



(a) Kirchoff: $\boxed{\sum_{i \in loop} V_i = 0}$

$$\mathcal{E} - IR - \frac{Q}{C} = 0$$

$$I = \frac{dQ}{dt}$$

$$\frac{\mathcal{E}}{R} - \frac{dQ}{dt} - \frac{Q}{RC} = 0$$

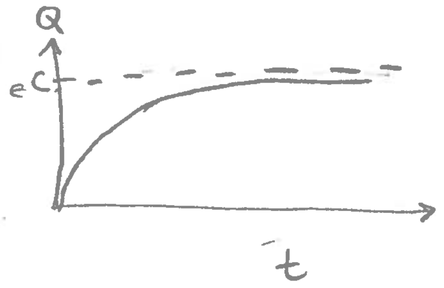
$$\frac{dQ}{dt} = \frac{\mathcal{E}}{R} - Q/RC$$

$$\int_0^Q \left(\frac{1}{\frac{\mathcal{E}}{R} - \frac{Q}{RC}} \right) dQ = \int_0^t dt$$

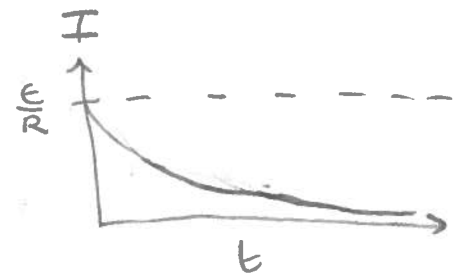
$$-RC \ln\left(\frac{\frac{\mathcal{E}}{R} - \frac{Q}{RC}}{\mathcal{E}/R} \right) = t$$

$$1 - \frac{Q}{\epsilon C} = e^{-t/RC}$$

$$Q(t) = \epsilon C\left(1 - e^{-t/RC}\right)$$



(b) $$I(t) = \frac{dQ}{dt} = \frac{\epsilon \cancel{C}}{R\cancel{C}}\, e^{-t/RC} = \frac{\epsilon}{R}\, e^{-t/RC}$$



(c) (i) $$U_{Res} = \int_0^\infty P\,dt = \int_0^\infty I^2 R\,dt = \left(\frac{\epsilon}{R}\right)^2 R \int_0^\infty e^{-2t/RC}\,dt$$

$$U_{Res} = \left(\frac{\epsilon}{R}\right)^2 R \left(-\frac{RC}{2}\right) e^{-2t/RC}\Big|_0^\infty = \frac{\epsilon^2 C}{2} = \frac{Q_{final}\,\epsilon}{2}$$

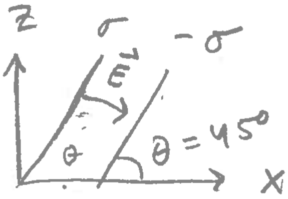
$Q_{final} = C\epsilon$

(ii) $$U_{cap} = \frac{Q_{fin}^2}{2C} = \frac{Q_{fin}\,\epsilon}{2}$$

(iii) $$U_{batt} = Q_f\,\epsilon$$

(iv) $U_{batt} = U_{cap} + U_{res}$. battery does total work $Q_f\,\epsilon$, and half is stored in field in Cap, and half is dissipated in the resister. Note: you can also check that energy is conserved at all times t.

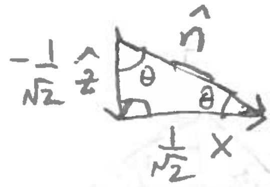
$\boxed{4}$



Gauss: $|E_{in}| = 4\pi k \sigma$

$\vec{E} \parallel \hat{n}$ of plates

$|E_{out}| = 0.$



$\therefore \hat{n} = \frac{1}{\sqrt{2}}\hat{x} - \frac{1}{\sqrt{2}}\hat{z}$

ⓐ $\vec{E}_{in} = \frac{4\pi k \sigma}{\sqrt{2}}(\hat{x} - \hat{z})$

$E_x = \frac{4\pi k \sigma}{\sqrt{2}}$ $\quad E_z = -\frac{4\pi k \sigma}{\sqrt{2}}$

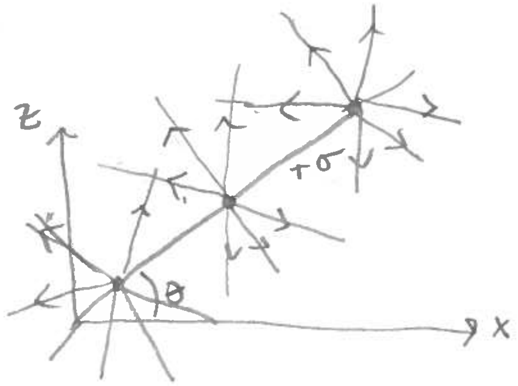
between plates

ⓑ $E'_x = E_x \leftarrow \parallel$ to boost

$E'_z = \gamma E_z = -\frac{4\pi k \sigma \gamma}{\sqrt{2}}$

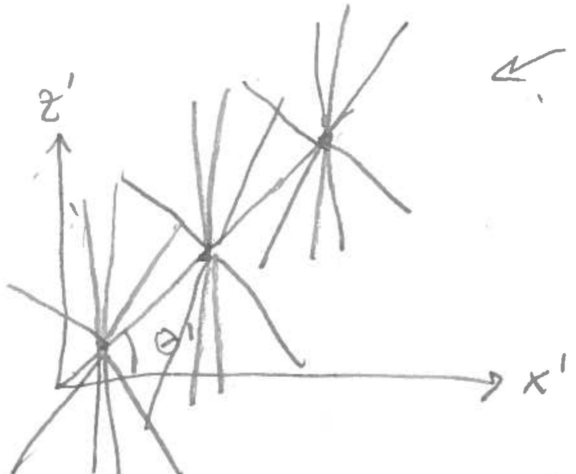
ⓒ

$\boxed{F}$



← components of E not normal to plate cancel.

$\boxed{F'}$

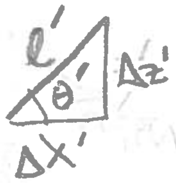


← components of E not normal to plate <u>do not</u> cancel!

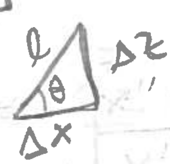
ⓓ $\Delta x' = \frac{1}{\gamma}\Delta x$

$\Delta z' = \Delta z$

$\boxed{F'}$



$\boxed{F}$



$$\theta'_p = \tan^{-1}\left(\frac{\Delta z'}{\Delta x'}\right) = \tan^{-1}\left(\gamma \overbrace{\left(\frac{\Delta z}{\Delta x}\right)}^{=1}\right) = \tan^{-1}(\gamma)$$

angle of plate in F'!

$$\theta'_E = \tan^{-1}\left(\frac{E'_z}{E'_x}\right) = -\tan^{-1}(\gamma)$$

For the plate and $\vec{E}$ field to be $\perp$,

$$\theta' - \theta'_E = 0$$

but from above we have

$$\theta' - \theta'_E = 2\tan^{-1}(\gamma)$$

only zero when $\gamma = 1$,

ie normal of plate & $\vec{E}$ field not $\perp$

in frame F'.