1. (20 points) The following values from the 8-point DFT of a length-8 real sequence $x[n]$ are known:

$X[0] = 3,\ X[2] = 0.5 - 4.5j,\ X[4] = 5,\ X[5] = 3.5 + 3.5j,\ X[7] = -2.5 - 7j.$

a) (5 points) Find the missing values $X[1]$, $X[3]$, $X[6]$.

b) (5 points) Evaluate $x[0]$.

c) (10 points) Find the 4-point DFT of the length-4 sequence $w[n]$ given by:

$$w[n] = x[n] + x[n+4] \quad n = 0, 1, 2, 3.$$

Hint: Derive a general formula that relates $W[k]$ to $X[k]$ so you don't have to calculate $x[n]$ and $w[n]$.

(Sam)

a). Because $x[n]$ is real,

$$X[1] = X^*[7] = -2.5 + 7j \tag{1}$$
$$X[3] = X^*[5] = 3.5 - 3.5j \tag{2}$$
$$X[6] = X^*[2] = 0.5 + 4.5j \tag{3}$$

b).

$$x[0] = \frac{1}{8}\sum_{k=0}^{7} X[k] \tag{4}$$
$$= \frac{1}{8}(3 + 2(-2.5) + 2(0.5) + 2(3.5) + 5) \tag{5}$$
$$= \frac{11}{8} \tag{6}$$

c).

$$W[k] = \sum_{n=0}^{3}(x[n] + x[n+4])e^{\frac{-2\pi}{4}kn} \tag{7}$$
$$= \sum_{n=0}^{7} x[n]e^{\frac{-2\pi}{4}kn} \tag{8}$$
$$= X[2k] \tag{9}$$

Thus

$$W[0] = X[0] = 3 \tag{10}$$
$$W[1] = X[2] = 0.5 - 4.5j \tag{11}$$
$$W[2] = X[4] = 5 \tag{12}$$
$$W[3] = X[6] = 0.5 + 4.5j \tag{13}$$

5. (20 points) When the input to an LTI is:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

the output is:

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n].$$

a) (10 points) Find the transfer function of $H(z)$ and indicate the region of convergence.

b) (5 points) Is the system causal? Is it stable?

b) (5 points) Write the difference equation that characterizes the system.

(Sam)

a). We have

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \quad (14)$$

$$= \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2} \quad (15)$$

$$= \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)}, \qquad \frac{1}{2} < |z| < 2 \quad (16)$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}} \quad (17)$$

$$= \frac{\frac{-3}{2}z}{(z-\frac{1}{2})(z-\frac{3}{4})}, \qquad |z| > \frac{3}{4}. \quad (18)$$

Then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z-2}{z-\frac{3}{4}} \quad (19)$$

$$= \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} \quad (20)$$

The ROC of $H(z)$ is either $|z| > \frac{3}{4}$ or $|z| < \frac{3}{4}$. Furthermore, the region of convergence of $Y(z)$ must include at least the intersection of $H(z)$ and $X(z)$, therefore the ROC of $H(z)$ must be

$$|z| > \frac{3}{4}. \quad (21)$$

b). Thus the system is stable (since the ROC includes the unit circle) and causal (since the ROC is the exterior of a circle).

c).

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1] \quad (22)$$