1. (20 points) The following values from the 8-point DFT of a length-8 real sequence $x[n]$ are known:

$X[0] = 3,\ X[2] = 0.5 - 4.5j,\ X[4] = 5,\ X[5] = 3.5 + 3.5j,\ X[7] = -2.5 - 7j.$

a) (5 points) Find the missing values $X[1]$, $X[3]$, $X[6]$.

b) (5 points) Evaluate $x[0]$.

c) (10 points) Find the 4-point DFT of the length-4 sequence $w[n]$ given by:

$$w[n] = x[n] + x[n+4] \quad n = 0, 1, 2, 3.$$

Hint: Derive a general formula that relates $W[k]$ to $X[k]$ so you don't have to calculate $x[n]$ and $w[n]$.

(Sam)

a). Because $x[n]$ is real,

$$X[1] = X^*[7] = -2.5 + 7j \tag{1}$$
$$X[3] = X^*[5] = 3.5 - 3.5j \tag{2}$$
$$X[6] = X^*[2] = 0.5 + 4.5j \tag{3}$$

b).

$$x[0] = \frac{1}{8}\sum_{k=0}^{7} X[k] \tag{4}$$
$$= \frac{1}{8}(3 + 2(-2.5) + 2(0.5) + 2(3.5) + 5) \tag{5}$$
$$= \frac{11}{8} \tag{6}$$

c).

$$W[k] = \sum_{n=0}^{3}(x[n] + x[n+4])e^{\frac{-2\pi}{4}kn} \tag{7}$$
$$= \sum_{n=0}^{7} x[n]e^{\frac{-2\pi}{4}kn} \tag{8}$$
$$= X[2k] \tag{9}$$

Thus

$$W[0] = X[0] = 3 \tag{10}$$
$$W[1] = X[2] = 0.5 - 4.5j \tag{11}$$
$$W[2] = X[4] = 5 \tag{12}$$
$$W[3] = X[6] = 0.5 + 4.5j \tag{13}$$

ANA

2. (20 points) The continuous-time signals $x(t)$ below are sampled to generate the corresponding discrete-time signals $x[n]$. Specify a choice for the sampling period T consistent with each pair. In addition, indicate whether the choice of T is unique. If not, specify a second choice of T.

a) (10 points) $x(t) = \sin(10\pi t) \quad \rightarrow \quad x[n] = \sin(\pi n/4)$

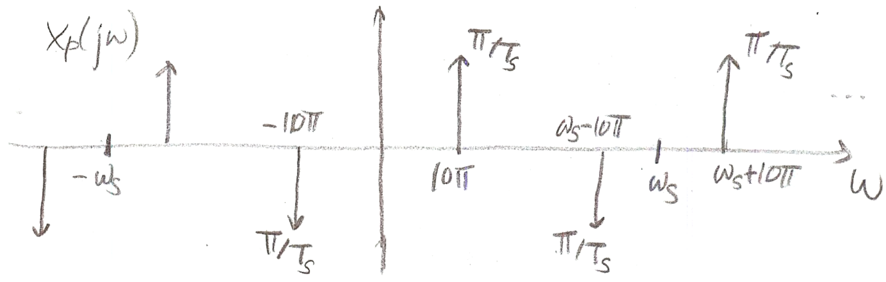
b) (10 points) $x(t) = \frac{\sin(10\pi t)}{10\pi t} \quad \rightarrow \quad x[n] = \frac{\sin(\pi n/2)}{\pi n/2}$

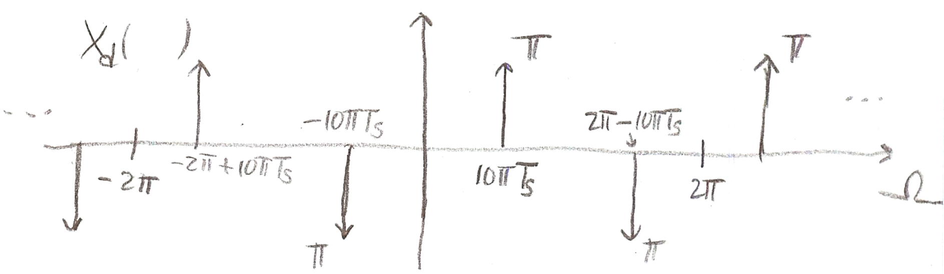
a) $x(t) = \sin(10\pi t)$

$X(j\omega) = \pi(-\delta(\omega+10\pi) + \delta(\omega - 10\pi))$

$X_p(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))$

$X_d(e^{j\Omega})\big|_{\Omega=\omega T_s} = X_p(j\omega)$





So, to have $x[n] = \sin(\pi n/4)$, we need

$10\pi T_s = \pi/4 \Rightarrow \boxed{T_s = 1/40}$

Or $x[n] = x(nT) = \sin(10\pi nT) \Rightarrow T = 1/40$

T_s is not unique: note that

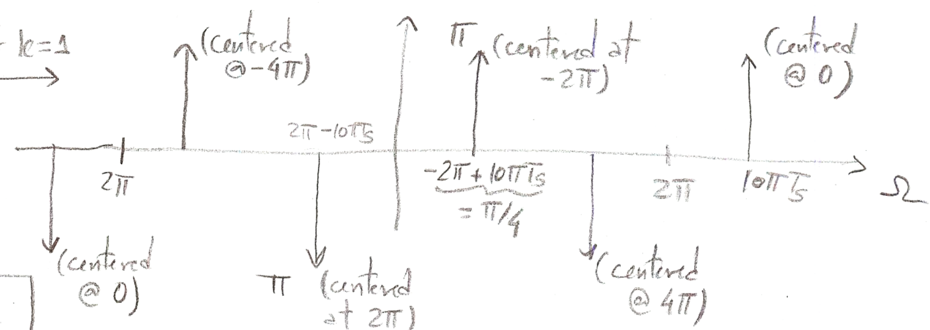
$\sin(\pi n/4) = \sin((\frac{\pi}{4} + 2\pi k)n)$

So, $\boxed{T_s = \frac{1}{40} + \frac{k}{5}}$, $k = \pm 1, 2, \ldots$

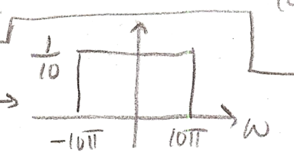
gives the same result

Or $x[n] = x(nT) = \sin(10\pi n T) =$
$= \sin((\pi/4 + 2\pi k)n)$

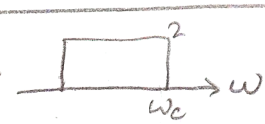
For $k=1$ $\longrightarrow$

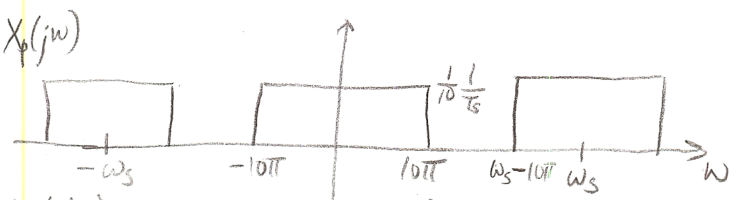


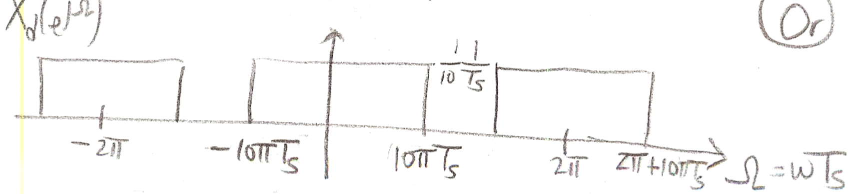
b) $x(t) = \frac{\sin(10\pi t)}{10\pi t} \longrightarrow$



Since $\frac{2\sin(\omega_c n)}{\pi n} \xrightarrow{\mathcal{F}}$







* for $x[n] = \frac{\sin(\pi n/2)}{\pi n/2}$, we need $10\pi T_s = \pi/2$

and also $1/10 \cdot 1/T_s = 2 \longrightarrow \boxed{T_s = 1/20}$

Or $x[n] = x(nT) = \frac{\sin(10\pi nT)}{10\pi nT} \Rightarrow T = 1/20$

Unique T since $x[n]$ is not periodic due to the denominator. From $X_d(e^{j\Omega})$ we can also see uniqueness of T.

ANA

3. (20 points) Consider the system below, where

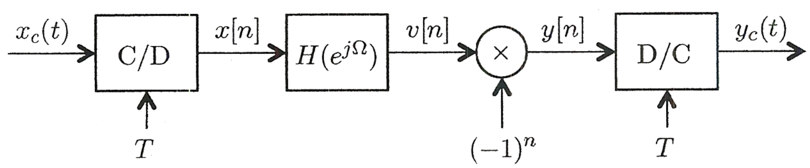
$$H(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < \pi/2 \\ 0, & \pi/2 < |\Omega| \leq \pi. \end{cases}$$

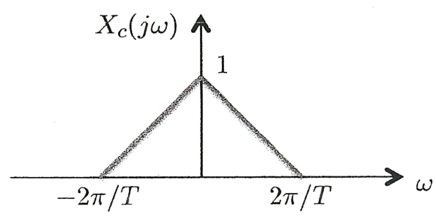
and assume the CTFT of the input, $X_c(j\omega)$, is as shown below.

a) (10 points) Sketch the DTFT for $x[n]$, $v[n]$ and $y[n]$.

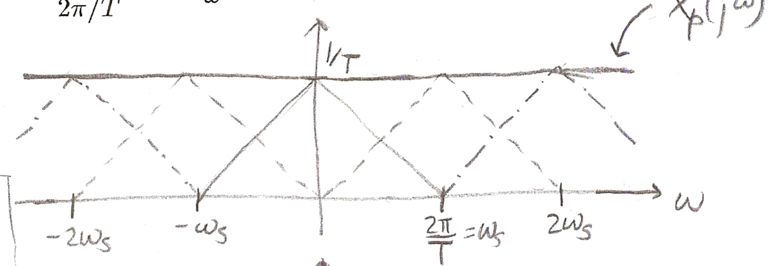
b) (5 points) Sketch the CTFT for the output, $Y_c(j\omega)$, assuming an ideal D/C converter.

c) (5 points) Sketch the magnitude $|Y_c(j\omega)|$ assuming, this time, a zero-order hold D/C converter.



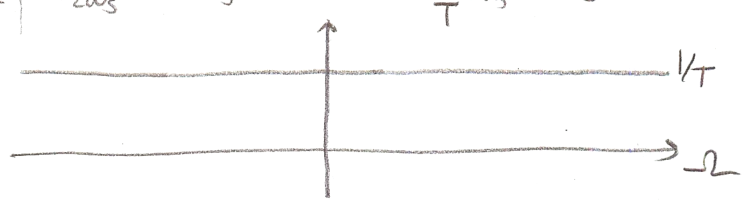


a) $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - \omega_s))$, where $\omega_s = \frac{2\pi}{T}$



So, $x[n] \longrightarrow X(e^{j\Omega})\big|_{\Omega = \omega T} = X_p(j\omega) \equiv \frac{1}{T} \; \forall \Omega$

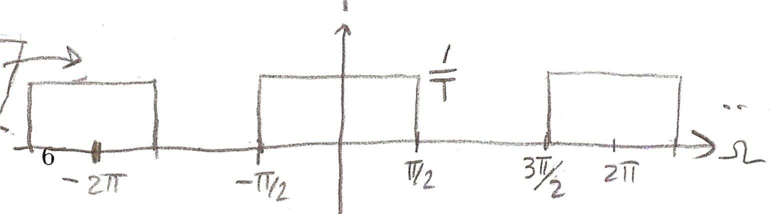
$\boxed{X(e^{j\Omega})}$



Since $X(e^{j\Omega}) \equiv \frac{1}{T} \; \forall \Omega$

$V(e^{j\Omega}) = \frac{1}{T} H(e^{j\Omega})$

$\boxed{V(e^{j\Omega})}$



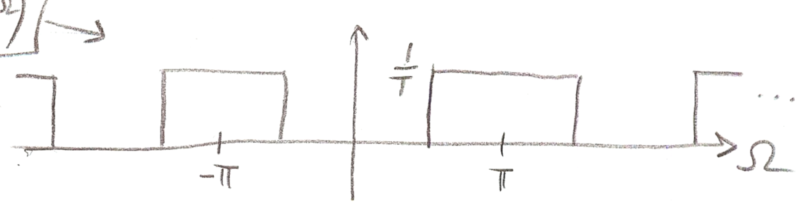
× Note that $(-1)^n = e^{j\pi n}$

and $y[n] = v[n] \times (-1)^n = e^{j\pi n} v[n]$

From frequency-shift property

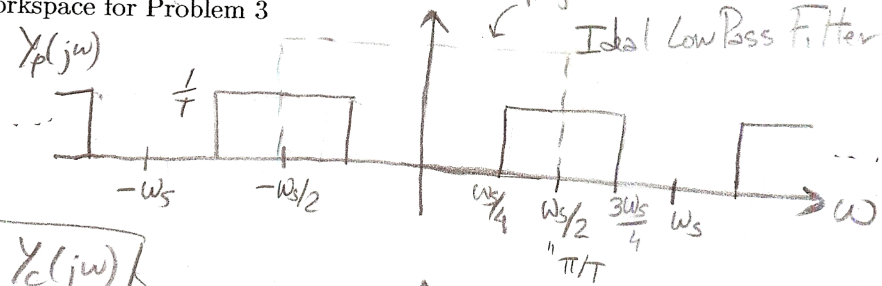
$Y(e^{j\Omega}) = V(e^{j(\Omega - \pi)})$

$\boxed{Y(e^{j\Omega})}$

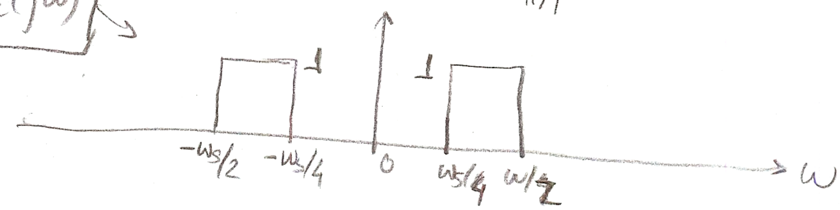


Additional workspace for Problem 3

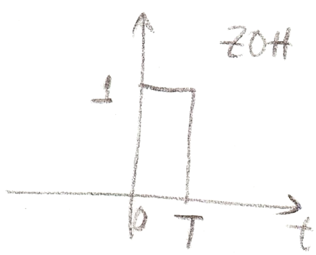
b) $Y_p(j\omega)\big|_{\omega=\Omega/T_s} = Y(e^{j\Omega})$

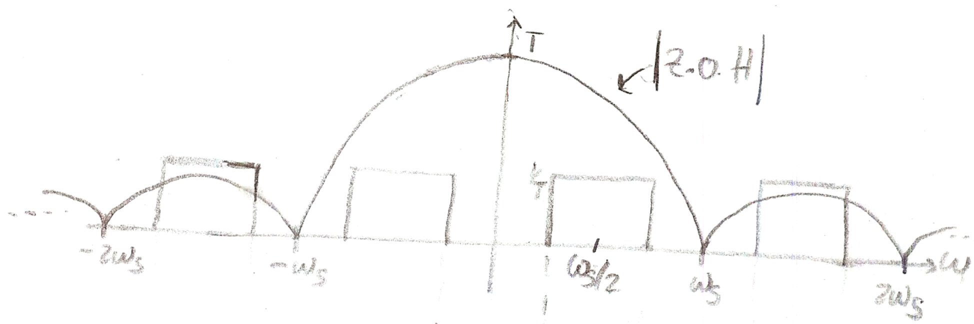


$\boxed{Y_c(j\omega)}$



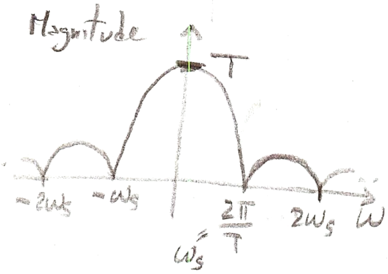
c)

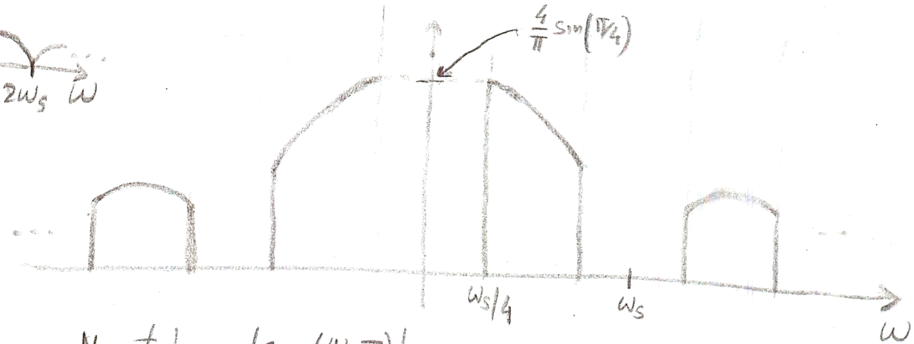




$\mathcal{F}\downarrow$

$e^{-j\omega T/2}\,\dfrac{\sin(\omega T/2)}{\omega/2}$





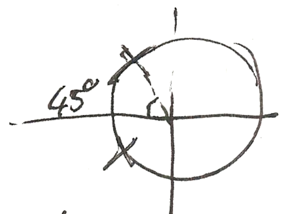
Magnitude is $\dfrac{|\sin(\frac{\omega}{2}T_s)|}{\omega T_s/2}$ when not zero.

4. a) (15 points) Specify the transfer function of a stable and causal LTI system whose frequency response has the magnitude:

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^4}}.$$

b) (5 points) Is the answer to part (a) unique? If not, specify another stable and causal LTI system whose frequency response has the same magnitude but a different phase.

a) This is a Butterworth filter (Lecture 15) with $\omega_c = 1$, $N = 2$. Therefore, poles:



$$s_{1,2} = -\frac{1}{\sqrt{2}} \mp j\frac{1}{\sqrt{2}}$$

$$H(s) = \frac{1}{\left(s+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}\right)\left(s+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)} = \frac{1}{s^2+\sqrt{2}s+1}.$$

(Alternatively, you can postulate a transfer function

$$H(s) = \frac{c}{s^2+as+b}$$

and solve for a, b, c to match the desired frequency response. You would have to choose $a>0$, $b>0$ to meet the stability requirement.)

Additional workspace for Problem 4.

b) Not unique. Multiply with a stable all-pass or simply multiply by -1 to keep the magnitude unchanged.

5. (20 points) When the input to an LTI is:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

the output is:

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n].$$

a) (10 points) Find the transfer function of $H(z)$ and indicate the region of convergence.

b) (5 points) Is the system causal? Is it stable?

b) (5 points) Write the difference equation that characterizes the system.

(Sam)

a). We have

$$\begin{aligned} X(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} && (14)\\ &= \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2} && (15)\\ &= \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)}, \qquad \frac{1}{2} < |z| < 2 && (16)\\ Y(z) &= \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}} && (17)\\ &= \frac{\frac{-3}{2}z}{(z-\frac{1}{2})(z-\frac{3}{4})}, \qquad |z| > \frac{3}{4}. && (18) \end{aligned}$$

Then

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{z-2}{z-\frac{3}{4}} && (19)\\ &= \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} && (20) \end{aligned}$$

The ROC of $H(z)$ is either $|z| > \frac{3}{4}$ or $|z| < \frac{3}{4}$. Furthermore, the region of convergence of $Y(z)$ must include at least the intersection of $H(z)$ and $X(z)$, therefore the ROC of $H(z)$ must be

$$|z| > \frac{3}{4}. \tag{21}$$

b). Thus the system is stable (since the ROC includes the unit circle) and causal (since the ROC is the exterior of a circle).

c).

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1] \tag{22}$$