Second Midterm Examination
Monday November 4 2013
Closed Books and Closed Notes

Question 1

A System of Two Particles

35 Points

Consider a particle of mass m_2 which is suspended below a cart of mass m_1 by a rod of negligible mass whose length ℓ changes with time: $\ell = \ell(t)$. The cart, which is free to move on a smooth horizontal track, is attached to a fixed point by a linear spring of stiffness K and unstretched length ℓ_0 and is under the influence of an applied force $\mathbf{P} = P_0 \cos(\omega t)\mathbf{E}_1$ (cf. Figure 1). The particles are under the influence of the respective gravitational forces $-m_1 g\mathbf{E}_3$ and $-m_2 g\mathbf{E}_3$.

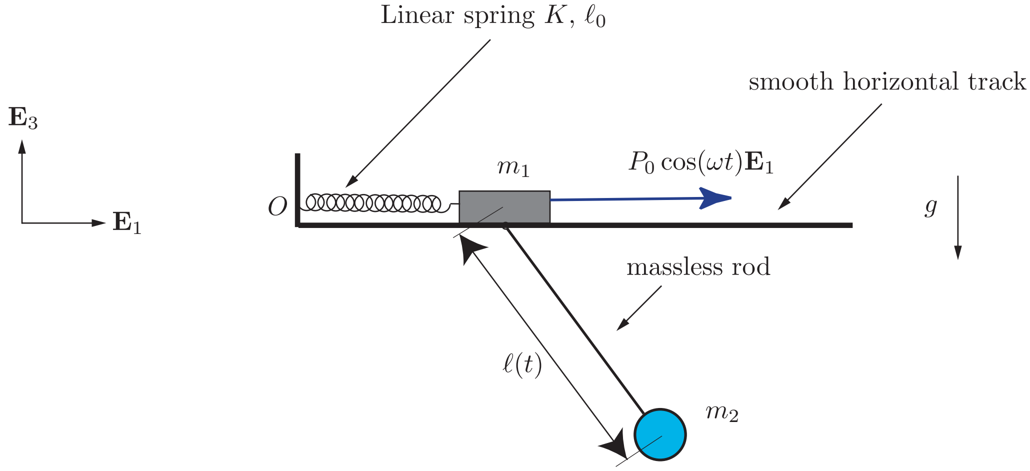


Figure 1: *A system of two particles. The particle of mass m_1 is free to move on a smooth horizontal track while a particle of mass m_2 is suspended by a rod of length $\ell(t)$ underneath. The mass of the rod is negligible.*

A Cartesian coordinate system is chosen to parameterize the motion of m_1 and a spherical polar coordinate system is chosen to parameterize $\mathbf{r}_2 - \mathbf{r}_1$:

$$\mathbf{r}_1 = (x + \ell_0 + c)\,\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3, \qquad \mathbf{r}_2 = (x + \ell_0 + c)\,\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3 + R\mathbf{e}_R. \tag{1}$$

Here, c is a constant such that when $x = 0$, the spring is unstretched.

(**a**) *(6 Points)* Compute the 12 vectors $\frac{\partial \mathbf{r}_i}{\partial q^K}$ where $q^1 = x, q^2 = \theta, q^3 = \phi, q^4 = y, q^5 = z$, and $q^6 = R$.

(**b**) *(6 Points)* What are the three constraints on the motion of the system of particles? Give prescriptions for the constraint forces $\mathbf{F}_{c_1}$ and $\mathbf{F}_{c_2}$ acting on the respective particles.

(**c**) *(3 Points)* If the kinetic energy of the system of particles has the representation

$$\begin{aligned} T \;=\; & \frac{m_1+m_2}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{m_2}{2}\left(\dot{R}^2+R^2\sin^2(\phi)\dot{\theta}^2+R^2\dot{\phi}^2\right) \\ & +m_2\dot{R}\left(\dot{x}\cos(\theta)\sin(\phi)+\dot{y}\sin(\theta)\sin(\phi)+\dot{z}\cos(\phi)\right) \\ & +m_2R\dot{\phi}\left(\dot{x}\cos(\theta)\cos(\phi)+\dot{y}\sin(\theta)\cos(\phi)-\dot{z}\sin(\phi)\right) \\ & +m_2R\sin(\phi)\dot{\theta}\left(-\dot{x}\sin(\theta)\;+\dot{y}\cos(\theta)\right), \end{aligned} \tag{2}$$

then what is the Lagrangian $\tilde{L} = \tilde{L}\left(x, \theta, \phi, \dot{x}, \dot{\theta}, \dot{\phi}, t\right)$ for the system of particles?

(**d**) *(3 Points)* Compute the following three summations:

$$\mathbf{F}_{ncon_1}\cdot\frac{\partial \mathbf{r}_1}{\partial q^k}+\mathbf{F}_{ncon_2}\cdot\frac{\partial \mathbf{r}_2}{\partial q^k}, \qquad k = 1,2,3, \tag{3}$$

where $\mathbf{F}_{ncon_\alpha}$ is the nonconservative force acting on the particle of mass m_α where $\alpha = 1, 2$.

(**e**) *(5 Points)* Show the combined power supplied by the constraint forces $\mathbf{F}_{c1}$ and $\mathbf{F}_{c2}$ on the system vanishes if $\dot{\ell}(t) = 0$.

(**f**) *(12 Points)* Show that the equations of motion of the system can be expressed in the form

$$\left[\begin{array}{ccc} (m_1+m_2) & -m_2\ell\sin(\theta)\sin(\phi) & m_2\ell\cos(\theta)\cos(\phi) \\ -m_2\ell\sin(\theta)\sin(\phi) & m_2\ell^2\sin^2(\phi) & 0 \\ m_2\ell\cos(\theta)\cos(\phi) & 0 & m_2\ell^2 \end{array}\right]\left[\begin{array}{c}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{array}\right]+\left[\begin{array}{c}f_1\\ f_2\\ f_3\end{array}\right]=\left[\begin{array}{c}P_0\cos(\omega t)\\ 0\\ 0\end{array}\right]. \tag{4}$$

For full credit, it is necessary to give complete expressions for $f_{1,2,3}$.

Question 2

A Classroom Demonstration

20 Points

Referring to Figure 2, a classroom demonstration of vibration phenomena consists of two particles of mass m_1 and m_2 which are connected by a nonlinear spring whose potential energy U_s is given by

$$U_s = \frac{K_1}{2}\left(\|\mathbf{r}_2 - \mathbf{r}_1\| - \ell_0\right)^2 + K_2\left(\frac{1}{\|\mathbf{r}_2 - \mathbf{r}_1\| - \ell_0}\right), \tag{5}$$

where $K_1 > 0$, K_2 and $\ell_0 > 0$ are constants, $\mathbf{r}_1$ is the position vector of the particle of mass m_1, and $\mathbf{r}_2$ is the position vector of the particle of mass m_2. In addition to the spring force, vertical gravitational forces act on the particles.

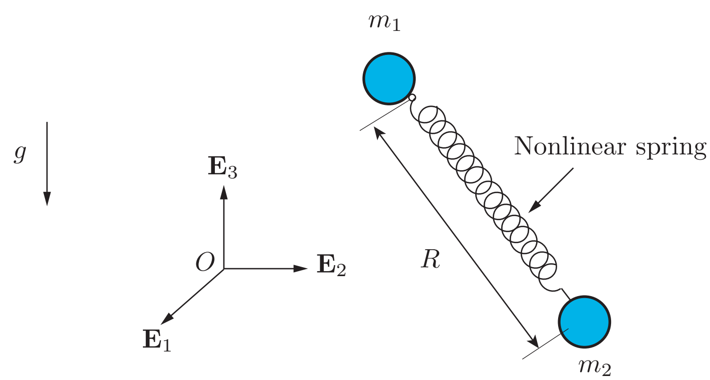


Figure 2: *A system of two particles. The particle of mass m_1 is constrained to move in a prescribed manner while the particle of mass m_2 is free to move in space.*

To analyze the system, a set of Cartesian coordinates x, y, and z are assigned to describe the position vector of the mass m_1 and a system of spherical polar coordinates R, ϕ, and θ are used to describe $\mathbf{r}_2 - \mathbf{r}_1$:

$$\mathbf{r}_1 = x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3, \qquad \mathbf{r}_2 = \mathbf{r}_1 + R\mathbf{e}_R. \tag{6}$$

We also choose

$$q^1 = R = \|\mathbf{r}_2 - \mathbf{r}_1\|, \quad q^2 = \theta, \quad q^3 = \phi, \quad q^4 = x, \quad q^5 = y, \quad q^6 = z. \tag{7}$$

In the sequel, we assume the motion of m_1 is completely prescribed:

$$\mathbf{r}_1 = f(t)\mathbf{E}_3. \tag{8}$$

(**a**) *(5 Points)* What is the constrained Lagrangian $\tilde{L}\left(R, \phi, \dot{R}, \dot{\phi}, \dot{\theta}, t\right)$ for the system of particles? [Feel free to use (2) to help with your solution.]

(**b**) *(5 Points)* Explain why Approach II can be used to establish the equations of motion.

(**c**) *(10 Points)* Show that the equations of motion for the system can be expressed in the form:

$$\begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2R^2\sin^2(\phi) & 0 \\ 0 & 0 & m_2R^2 \end{bmatrix} \begin{bmatrix} \ddot{R} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -m_2\left(g + \ddot{f}\right)\cos(\phi) \\ 0 \\ m_2\left(g + \ddot{f}\right)R\sin(\phi) \end{bmatrix}. \tag{9}$$

For full credit, supply expressions for f_1, f_2, and f_3.

Notes on Spherical Polar Coordinates

Recall that the spherical polar coordinates $\{R, \phi, \theta\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}\,, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right)\,, \quad \phi = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right).$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

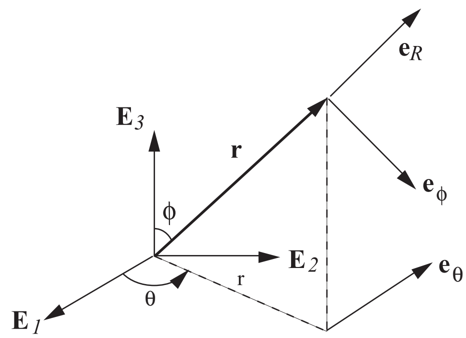


Figure 3: Spherical polar coordinates

For the coordinate system $\{R, \phi, \theta\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_R\,, \quad \mathbf{a}_2 = R\mathbf{e}_\phi\,, \quad \mathbf{a}_3 = R\sin(\phi)\mathbf{e}_\theta.$$

In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_R\,, \quad \mathbf{a}^2 = \frac{1}{R}\mathbf{e}_\phi\,, \quad \mathbf{a}^3 = \frac{1}{R\sin(\phi)}\mathbf{e}_\theta.$$

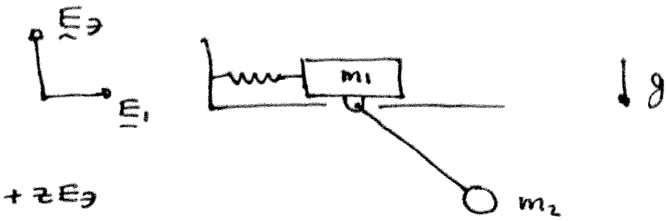
For a particle of mass m which is unconstrained, the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_O$ and kinetic energy T of the particle are

$$\begin{aligned} \mathbf{G} &= m\dot{R}\mathbf{a}_1 + m\dot{\phi}\mathbf{a}_2 + m\dot{\theta}\mathbf{a}_3\,, \\ \mathbf{H}_O &= mR^2\left(\dot{\phi}\mathbf{e}_\theta - \dot{\theta}\sin(\phi)\mathbf{e}_\phi\right)\,, \\ T &= \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2(\phi)\dot{\theta}^2\right). \end{aligned}$$

The gradient of a function $U(R, \theta, \phi)$ has the representation

$$\nabla u = \frac{\partial u}{\partial R}\mathbf{e}_R + \frac{\partial u}{\partial \theta}\frac{1}{R\sin(\phi)}\mathbf{e}_\theta + \frac{1}{R}\frac{\partial u}{\partial \phi}\mathbf{e}_\phi.$$

QUESTION 1



$$\underline{r}_1 = (x + c + l_0)\underline{E}_1 + y\underline{E}_2 + z\underline{E}_3$$

$$\underline{r}_2 = \underline{r}_1 + R\underline{e}_R$$

(a)

	x	θ	ϕ	y	z	R
$\frac{\partial \underline{r}_1}{\partial q^K}$	$\underline{E}_1$	$\underline{0}$	$\underline{0}$	$\underline{E}_2$	$\underline{E}_3$	$\underline{0}$
$\frac{\partial \underline{r}_2}{\partial q^K}$	$\underline{E}_1$	$R\sin\phi\,\underline{e}_\theta$	$R\underline{e}_\phi$	$\underline{E}_2$	$\underline{E}_3$	$\underline{e}_R$

(b)

$$\psi_1 = \underline{r}_1 \cdot \underline{E}_2 = 0$$

$$\psi_2 = \underline{r}_1 \cdot \underline{E}_3 = 0$$

$$\psi_3 = \|\underline{r}_2 - \underline{r}_1\| - l = 0$$

$$\underline{F}_{c_1} = \lambda_1 \underline{E}_2 + \lambda_2 \underline{E}_3 + \lambda_3 \underline{e}_R$$

$$\underline{F}_{c_2} = \lambda_3 \underline{e}_R$$

$\lambda_1 \underline{E}_2 + \lambda_2 \underline{E}_3$ is the normal force acting on m_1

$\lambda_3 \underline{e}_R$ is the tension force in the rod.

(c)

$$\tilde{L} = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}\left(l^2\dot{\phi}^2 + l^2\dot{\theta}^2\sin^2\phi\right)$$

$$+ m_2 \dot{l}\dot{x}\cos\theta\sin\phi + m_2 l\dot{\phi}\dot{x}\cos\theta\cos\phi - m_2 l\sin\phi\sin\theta\,\dot{x}\dot{\theta}$$

$$- \tilde{U}$$

$$\tilde{U} = \frac{1}{2}Kx^2 + m_2 g l\cos\phi$$

(d)

$$\underset{\sim}{F}_{ncon_1} \cdot \frac{\partial \underset{\sim}{r}_1}{\partial x} + \underset{\sim}{F}_{ncon_2} \cdot \frac{\partial \underset{\sim}{r}_2}{\partial x} = \left(F_{ncon_1} + F_{ncon_2} \right) \cdot \underset{\sim}{E}_1$$

$$= \left(\underset{\sim}{F}_{c1} + P_0 \cos \omega t \, \underset{\sim}{E}_1 + \underset{\sim}{F}_{c2} \right) \cdot \underset{\sim}{E}_1$$

$$= P_0 \cos \omega t$$

$$\underset{\sim}{F}_{ncon_1} \cdot \frac{\partial \underset{\sim}{r}_1}{\partial \theta} + F_{ncon_2} \cdot \frac{\partial \underset{\sim}{r}_2}{\partial \theta} = \underset{\sim}{F}_{ncon_2} \cdot R \sin\phi \, \underset{\sim}{e}_\theta$$

$$= \underset{\sim}{F}_{c2} \cdot R \sin\phi \, e_\theta$$

$$= 0$$

$$F_{ncon_2} \cdot \frac{\partial \underset{\sim}{r}_1}{\partial \phi} + \underset{\sim}{F}_{ncon_2} \cdot \frac{\partial \underset{\sim}{r}_2}{\partial \phi} = F_{ncon_2} \cdot R \underset{\sim}{e}_\phi$$

$$= \underset{\sim}{F}_{c2} \cdot R \underset{\sim}{e}_\phi$$

$$= 0.$$

(e)

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^K}\right) - \frac{\partial \tilde{L}}{\partial q^K} = \underset{\sim}{F}_{ncon_1} \cdot \frac{\partial \underset{\sim}{r}_1}{\partial q^K} + \underset{\sim}{F}_{ncon_2} \cdot \frac{\partial \underset{\sim}{r}_2}{\partial q^K} \quad \text{where } K = 1, 2, 3$$

$$\frac{d}{dt}\Big(\frac{\partial \tilde{L}}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2 \dot{\ell} \cos\theta \sin\phi + m_2 \ell \dot{\phi} \cos\theta \cos\phi - m_2 \ell \sin\phi \sin\theta \, \dot{\theta} \Big)$$

$$+ Kx = P_0 \cos\omega t$$

$$\frac{d}{dt}\Big(\frac{\partial \tilde{L}}{\partial \dot{\theta}} = m_2 \ell^2 \dot{\theta} \sin^2\phi - m_2 \ell \sin\phi \sin\theta \, \dot{x}\Big)$$

$$+ m_2 \dot{\ell} \dot{x} \sin\theta \sin\phi + m_2 \ell \dot{\phi} \dot{x} \sin\theta \cos\phi + m_2 \ell \sin\phi \cos\theta \, \dot{x} \dot{\theta} = 0$$

$$\frac{d}{dt}\left(\Big(\frac{\partial \tilde{L}}{\partial \dot{\phi}}\right) = m_2 \ell^2 \dot{\phi} + m_2 \ell \dot{x} \cos\theta \cos\phi \Big) - m_2 \ell^2 \dot{\theta}^2 \sin\phi \cos\phi + m_2 \dot{\ell} \dot{x} \cos\theta \cos\phi$$

$$+ m_2 \ell \dot{\phi} \dot{x} \sin\phi \cos\theta + m_2 \ell \dot{x} \dot{\theta} \cos\phi \sin\theta - m_2 g \ell \sin\phi = 0$$

Expanding and rearranging

$$\begin{bmatrix} m_1+m_2 & -m_2 l \sin\phi \sin\theta & m_2 l \cos\theta \cos\phi \\ -m_2 l \sin\phi \sin\theta & m_2 l^2 \sin^2\phi & 0 \\ m_2 l \cos\theta \cos\phi & 0 & m_2 l^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} P_0 \cos\omega t \\ 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 = Kx + \frac{d}{dt}\left(m_2 l \cos\theta \cos\phi \right) \dot{\phi} - \frac{d}{dt}\left(m_2 l \sin\phi \sin\theta \right) \dot{\theta}$$

$$+ \frac{d}{dt}\left(m_2 \dot{l} \cos\theta \sin\phi \right)$$

$$\alpha_2 = \frac{d}{dt}\left(m_2 l^2 \sin^2\phi \right) \dot{\theta}$$ (several cancellations occur in this equation)

$$\alpha_3 = -m_2 l^2 \dot{\theta}^2 \sin\phi \cos\phi - m_2 g l \sin\phi$$

(5) $P = \underline{F}_{c_1} \cdot \underline{V}_1 + \underline{F}_{c_2} \cdot \underline{V}_2 = (\lambda_1 \underline{E}_2 + \lambda_2 \underline{E}_3 + \lambda_3 \underline{e}_R) \cdot \underline{V}_1$

$$+ \lambda_3 \underline{e}_R \cdot \underline{V}_2$$

$$= \lambda_3 \underline{e}_R \cdot (\underline{V}_2 - \underline{V}_1)$$

$$= \lambda_3 \dot{l}$$

Hence P vanishes if $\dot{l} = 0$ which was to be shown.

QUESTION 2

(a)
$$\tilde{\mathcal{L}} = \frac{m_2}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\dot{\theta}^2\sin^2\phi\right) + \dot{f}\left(m_2\dot{R}\cos\phi - m_2 R\dot{\phi}\sin\phi\right) - m_2 g R\cos\phi - \frac{K_1}{2}\left(R-\ell_0\right)^2 - \frac{K_2}{R-\ell_0}$$

(b) Approach II can be used for this problem because

(i) The integrable constraints can all be expressed in terms of a single coordinate

$$\psi_1 = \underline{r}_1 \cdot \underline{E}_1 = 0 \iff q^4 = 0$$

$$\psi_2 = \underline{r}_1 \cdot \underline{E}_2 = 0 \iff q^5 = 0$$

$$\psi_3 = \underline{r}_1 \cdot \underline{E}_3 - f = 0 \iff q^6 - f = 0$$

(ii) The constraint forces $\underline{F}_{c_1}$ acting on the system can be prescribed using Lagrange's prescription.

(c)
$$\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{R}} = m_2\dot{R} + m_2\dot{f}\cos\phi\right) - m_2 R\dot{\phi}^2 - m_2 R\dot{\theta}^2\sin^2\phi + m_2\dot{f}\dot{\phi}\sin\phi + m_2 g\cos\phi + K_1(R-\ell_0) - \frac{K_2}{(R-\ell_0)^2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{\theta}} = m_2 R^2\dot{\theta}\sin^2\phi\right) - \left(\frac{\partial\tilde{L}}{\partial\theta} = 0\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{\phi}} = m_2 R^2\dot{\phi} - m_2 R\dot{f}\sin\phi\right) - \left(\frac{\partial\tilde{L}}{\partial\phi} = m_2 R^2\dot{\theta}^2\sin\phi\cos\phi - m_2\dot{R}\dot{f}\sin\phi - m_2 R\dot{f}\dot{\phi}\cos\phi + m_2 g R\sin\phi\right) = 0$$

Expanding and combining terms

$$
\begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2 R^2 \sin^2\phi & 0 \\ 0 & 0 & m_2 R^2 \end{bmatrix}
\begin{bmatrix} \ddot{R} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix}
+
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
=
\begin{bmatrix} m_2 (g + \ddot{f}) \cos\phi \\ 0 \\ m_2 (g+\ddot{f}) R \sin\phi \end{bmatrix}
$$

mass matrix terms can be found by inspection from $\tilde{T}$

(combined with $m_2 g \cos\phi$)

$$
\begin{aligned}
\alpha_1 = & \; \cancel{- m_2 \dot{f} \dot{\phi} \sin\phi} + \cancel{m_2 \ddot{f} \cos\phi} \\
& - m_2 R \dot{\phi}^2 - m_2 R \dot{\theta}^2 \sin^2\phi + \cancel{m_2 \dot{f} \dot{\phi} \sin\phi} + K_1 (R - l_0) - \frac{K_2}{(R - l_0)^2}
\end{aligned}
$$

$$\cancel{\alpha_3} = \cancel{2 m_2 R \dot{R} \dot{\phi}} \quad \cancel{- m_2 \dot{R} \dot{f}}$$

$$\alpha_2 = 2 m_2 R \dot{R} \dot{\theta} \sin^2\phi + 2 m_2 R^2 \dot{\theta} \sin\phi \cos\phi \, \dot{\phi}$$

$$
\begin{aligned}
\alpha_3 = & \; 2 m_2 R \dot{R} \dot{\phi} - m_2 \dot{R} \dot{f} \sin\phi - m_2 R \dot{f} \dot{\phi} \cos\phi \\
& - m_2 R^2 \dot{\theta}^2 \sin\phi \cos\phi + m_2 \dot{R} \dot{f} \sin\phi + m_2 R \dot{f} \dot{\phi} \cos\phi \\
= & \; 2 m_2 R \dot{R} \dot{\phi} - m_2 R^2 \dot{\theta}^2 \sin\phi \cos\phi
\end{aligned}
$$

The resulting equations of motion are identical to those that would be obtained if we let gravity $g \rightarrow g + \ddot{f}$ and set $\underline{\Gamma}_1 = \underline{0}$.