EXAM # 2 SOLUTIONS

$$h[n] = \begin{cases} \frac{1}{3} & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

PROBLEM 1

(GOELA)

i.e. $h[n] = \frac{1}{3}\left[\delta[n] + \delta[n-1] + \delta[n-2]\right]$

PART (A) Calculate + Sketch Phase $\measuredangle H(e^{j\omega})$.

Both $|H(e^{j\omega})|$ and $\measuredangle H(e^{j\omega})$ are periodic/repeated every 2π.

$$H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \frac{1}{3}(1 + e^{-j\omega} + e^{-2j\omega})$$

$$= \frac{1}{3} e^{-j\omega}(e^{j\omega} + 1 + e^{-j\omega})$$

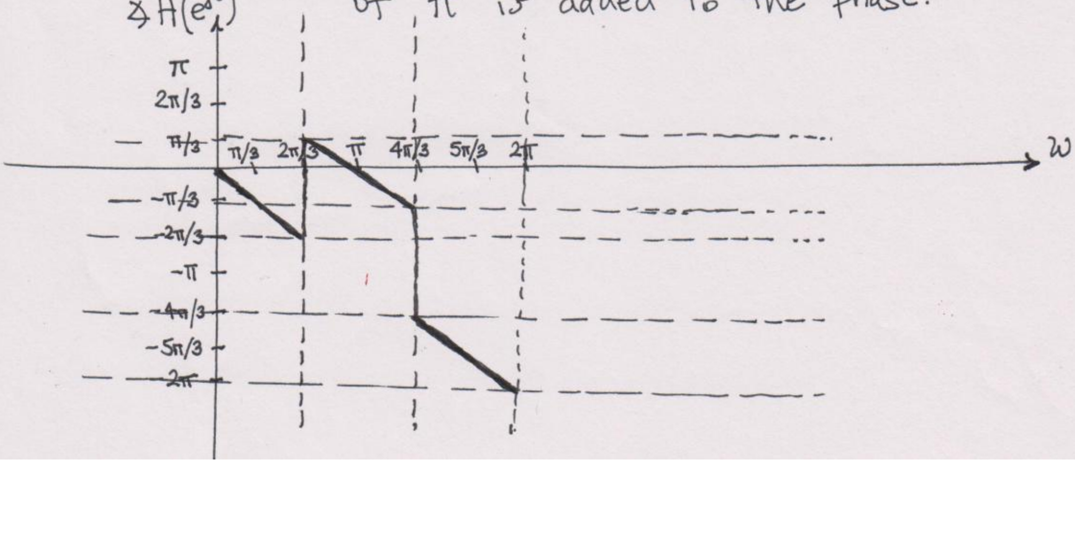
$$= e^{-j\omega}\left[\frac{1}{3}(1 + 2\cos(\omega))\right]$$

$$|H(e^{j\omega})| = \left|\frac{1}{3}(1 + 2\cos(\omega))\right|$$

FOR ONE PERIOD $\omega \in [0, 2\pi)$:

$$\boxed{\measuredangle H(e^{j\omega}) = -\omega + \pi \cdot \mathbb{1}_{\left[\frac{2\pi}{3} < \omega < \frac{4\pi}{3}\right]} + 2\pi k \quad \text{where } k \in \mathbb{Z} \text{ is integer.}}$$

Note: When $[1 + 2\cos(\omega)] < 0$, a flip of π is added to the phase.



PROBLEM 1
PART (B)

$$h[n] = \frac{1}{3}\left(\delta[n] + \delta[n-1] + \delta[n-2]\right)$$

$$H(e^{j\omega}) = \left[\frac{1}{3}(1 + 2\cos(\omega))\right] e^{-j\omega}$$

$\boxed{\text{"Generalized Linear Phase"}}$ means $H(e^{j\omega}) = A(e^{j\omega}) e^{-j\alpha\omega + j\beta}$.

Therefore, the system has GLP, i.e.

$$\boxed{\begin{aligned} A(e^{j\omega}) &= \frac{1}{3}(1 + 2\cos(\omega)) \\ \alpha &= 1 \\ \beta &= 0 \end{aligned}}$$

The group delay is constant.

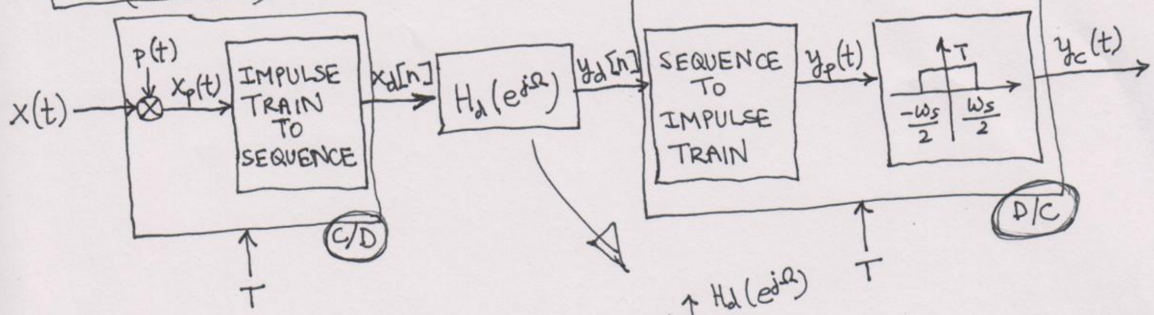
$$\boxed{\tau_g(\omega) = \frac{-d}{d\omega} \measuredangle H(e^{j\omega}) = 1.}$$

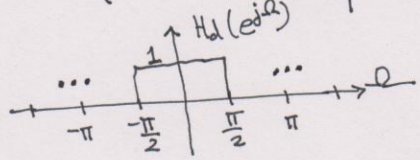
So, in this sense, with $\tau_g(\omega) = 1$, the system has a "linear phase" but as introduced in lecture notes, since $A(e^{j\omega})$ is not strictly positive, there are flips of π in the phase.

In this sense, the system is not strictly linear phase.

EXAM #2

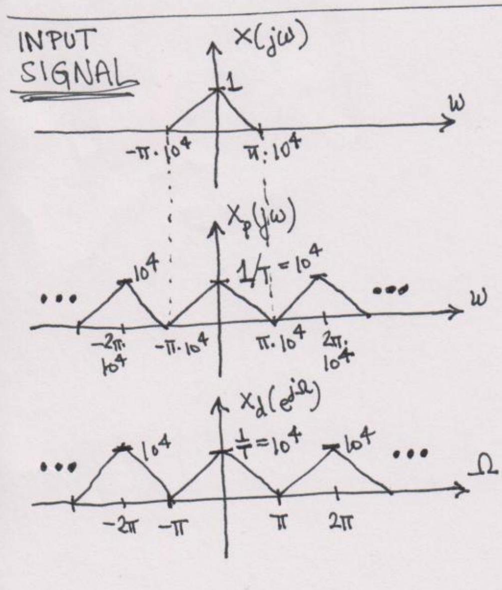
PROBLEM 2
PART (A)
(GOELA)

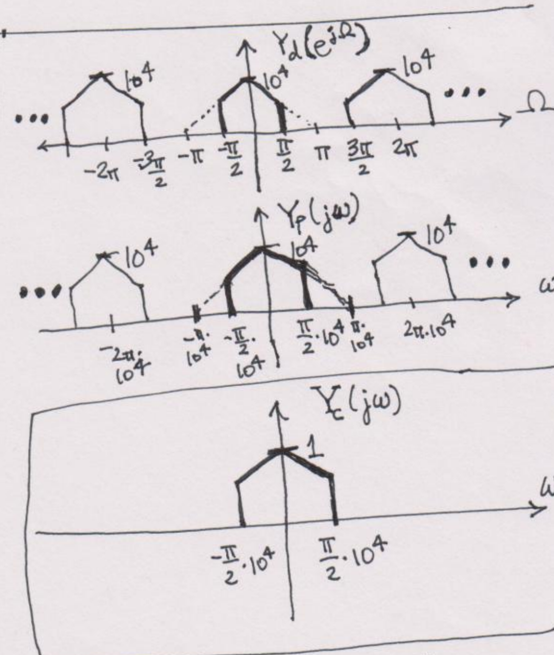




$T = 10^{-4}$ sec.

$\omega_s = \frac{2\pi}{T} = 2\pi \cdot 10^4$ rad/sec.





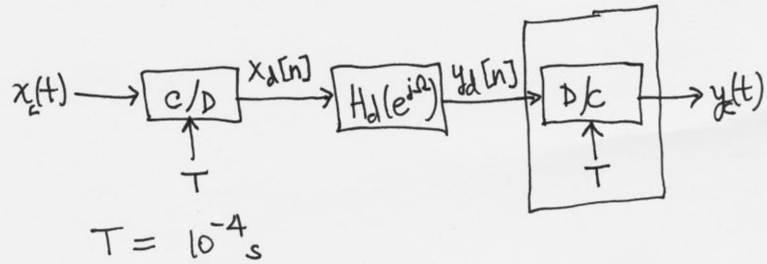
SKETCH OF $Y_c(j\omega)$ SPECTRUM.

OUTPUT SIGNAL

EXAM # 2

PROBLEM 2
PART (B)

(GOELA)



$T = 10^{-4}$ s

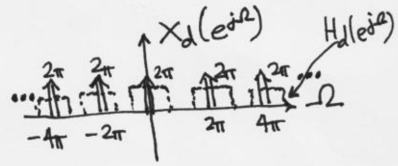
$\omega_s = \frac{2\pi}{T} = 2\pi \cdot 10^4$ rad/sec.

INPUT: $\boxed{x_c(t) = \cos(2\pi \cdot 10^4 t)}$

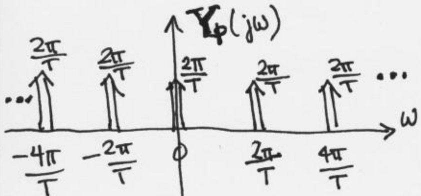
$x_d[n] = x_c(nT) = \cos(2\pi \cdot 10^4 \cdot 10^{-4} \cdot n) = \cos(2\pi n) = 1$

$\boxed{x_d[n] = 1}$

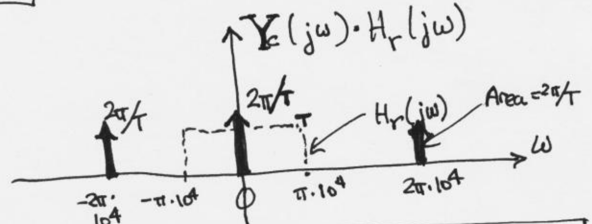
$$\boxed{X_d(e^{j\Omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - 2\pi l)}$$

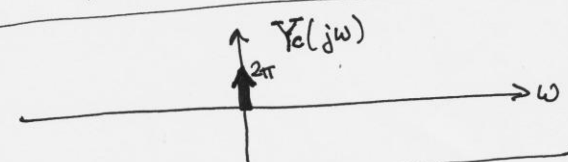


$\boxed{Y_d(e^{j\Omega}) = X_d(e^{j\Omega})}$ LOW-FREQUENCY PASSED THROUGH.



where $\frac{2\pi}{T} = 2\pi \cdot 10^4$ rad/sec.





SOLUTION:

$Y_c(j\omega) = 2\pi\,\delta(\omega)$

$y(t) = 1$

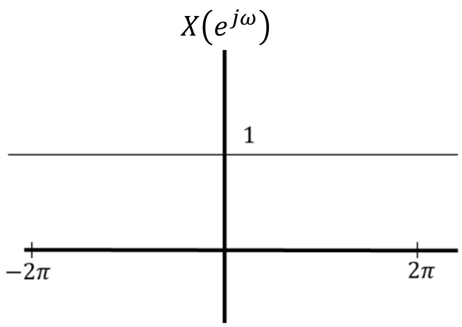
Question 3

For notation, refer to lecture 13, pages 3 to 5.

Part a)

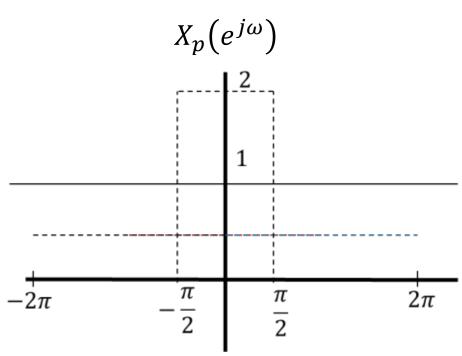
In the frequency domain, the original signal $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = 1 \ \forall\omega$$

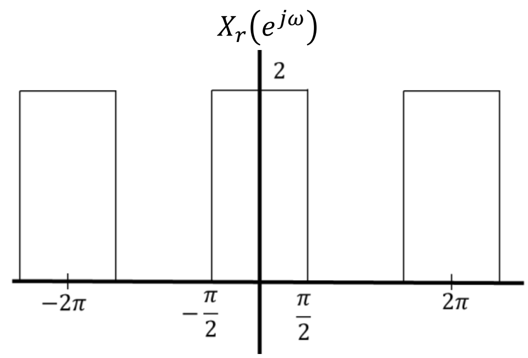


Then, we select every second sample and discard the rest to get $x_p[n]$. The resulting signal is still $\delta[n]$, so $X_p(e^{j\omega}) = 1 \ \forall\omega$.

One can also obtain this by considering $X(e^{j\omega})$ for $\omega \in [-\pi, \pi]$. First scale the height of $X(e^{j\omega})$ by $\frac{1}{2}$, and then repeat this segment every $\frac{2\pi}{N} = \pi$ (the colored, dotted horizontal lines in the plot below). Then, the overlapping parts add up and we get back $X_p(e^{j\omega}) = 1 \ \forall\omega$. A common mistake is to not add these overlapping parts.



From here we could proceed to determine $X_b(e^{j\omega})$, but there is no need because during upsampling we will convert $X_b(e^{j\omega})$ back to $X_p(e^{j\omega})$ anyway. Thus, we can just directly apply the ideal low-pass filter to $X_p(e^{j\omega})$. The filter is a box with width $\frac{\pi}{N} = \frac{\pi}{2}$ and height $N = 2$, as shown in the figure above.



Applying the filter, we get $X_r(e^{j\omega})$, which is NOT the original signal. The corresponding time domain signal is calculated in lecture 9, page 4. Note that in this problem the height of the box is 2, thus

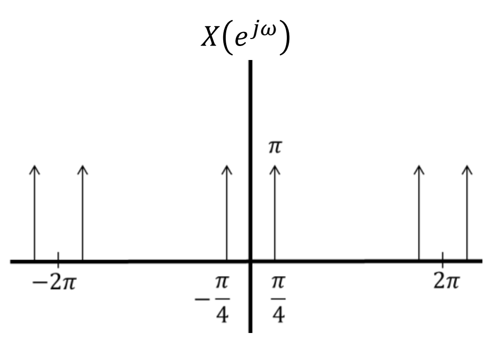
$$x_r[n] = \frac{2\sin\left(\frac{\pi}{2}n\right)}{\pi n} = \frac{\sin\left(\frac{\pi}{2}n\right)}{\frac{\pi}{2}n}$$

Part b)

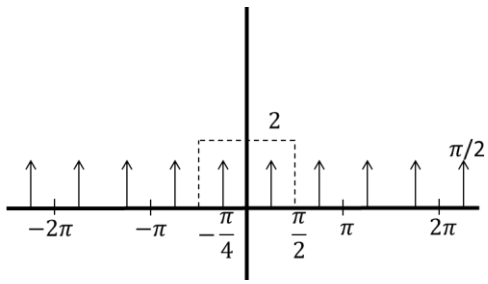
Here, we can use the same method as the method used in part a) by noting that

$$\cos\left(\frac{\pi n}{4}\right) = \frac{1}{2}\left(e^{\frac{j\pi n}{4}} + e^{-\frac{j\pi n}{4}}\right)$$
$$\leftrightarrow \pi\delta\left(\omega - \frac{\pi}{4}\right) + \pi\delta\left(\omega + \frac{\pi}{4}\right)$$

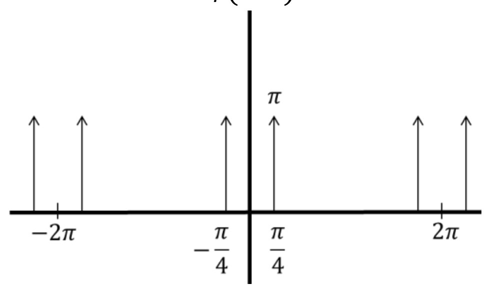
After the same steps in part a), we will discover that $x_r[n] = \cos\left(\frac{\pi n}{4}\right) = x[n]$:



$X_p(e^{j\omega})$



$X_r(e^{j\omega})$



Alternatively, we can also note that $\cos\left(\frac{\pi n}{4}\right)$ can be obtained by sampling the continuous time signal $\cos\left(\frac{\pi t}{4}\right)$ with sampling period 1. The corresponding sampling angular frequency 2π. After downsampling by a factor of 2, we would have that a new sampling rate $\omega_s = \pi$.

Now, the bandwidth of $\cos\left(\frac{\pi t}{4}\right)$ is $\omega_M = \frac{\pi}{4}$, so we have that $\pi = \omega_s > 2\omega_M = \frac{\pi}{2}$. Therefore, an ideal reconstruction filter will be able to recover the original CT function $\cos\left(\frac{\pi t}{4}\right)$. In particular, the samples in $x[n] = \cos\left(\frac{\pi n}{4}\right)$ can be recovered.

Question 4

Part a)

By adding and subtracting $s^2 + 2s + 2$ in the numerator, we get

$$
\begin{aligned}
X(s) &= \frac{2s+3}{(s-1)(s^2+2s+2)} \\
&= \frac{s^2+2s+2-(s^2+2s+2)+2s+3}{(s-1)(s^2+2s+2)} \\
&= \frac{1}{s-1} - \frac{s^2-1}{(s-1)(s^2+2s+2)} \\
&= \frac{1}{s-1} - \frac{(s+1)(s-1)}{(s-1)(s^2+2s+2)} \\
&= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+1} \\
&\leftrightarrow -e^{t}u(-t) - e^{-t}\cos(t)\,u(t)
\end{aligned}
$$

Note that we’ve chosen $\frac{1}{s-1} \leftrightarrow -e^{t}u(-t)$ instead of $\leftrightarrow e^{t}u(t)$ to ensure that $x(t)$ is absolutely integrable.

We could have also simplified $X(s)$ using partial fractions as follows:

$$\frac{2s+3}{(s-1)(s^2+2s+2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2}$$

Alternatively, we could have also done partial fraction with three different denominators, although there is a bit more work involved:

$$
\begin{aligned}
X(s) &= \frac{2s+3}{(s-1)(s^2+2s+2)} \\
&= \frac{2s+3}{(s-1)(s+1+j)(s+1-j)} \\
&= \frac{A}{s-1} + \frac{B}{s+1+j} + \frac{C}{s+1-j} \\
&= \frac{A(s+1+j)(s+1-j) + B(s-1)(s+1-j) + C(s-1)(s+1+j)}{(s-1)(s+1+j)(s+1-j)}
\end{aligned}
$$

$$
\begin{aligned}
&s = -1-j \\
&\Rightarrow 2(-1-j)+3 = B(-1-j-1)(-1-j+1-j) \\
&\Rightarrow 1-2j = B(-2-j)(-2j) \\
&\Rightarrow 1-2j = -2B(1-2j) \\
&\Rightarrow B = -\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&s = 1 \\
&\Rightarrow 5 = A(1+1+j)(1+1-j) \\
&\Rightarrow 5 = A(4+1) \\
&\Rightarrow A = 1
\end{aligned}
$$

$$
\begin{aligned}
&s = -1+j \\
&\Rightarrow 2(-1+j)+3 = C(-1+j-1)(-1+j+1+j) \\
&\Rightarrow 1+2j = C(-2+j)(2j) \\
&\Rightarrow 1+2j = -2C(2j+1) \\
&\Rightarrow C = -\frac{1}{2}
\end{aligned}
$$

Thus

$$X(s) = \frac{1}{s-1} - \frac{1}{2}\left(\frac{1}{s+1+j} + \frac{1}{s+1-j}\right)$$

$$
\begin{aligned}
x(t) &= -e^{t}u(-t) - \frac{1}{2}\left(e^{(-1-j)t}u(t) + e^{(-1+j)t}u(t)\right) \\
&= -e^{t}u(-t) - \frac{1}{2}\left(e^{-t}e^{-jt} + e^{-t}e^{jt}\right)u(t) \\
&= -e^{t}u(-t) - e^{-t}\frac{1}{2}\left(e^{-jt} + e^{jt}\right)u(t) \\
&= -e^{t}u(-t) - e^{-t}\cos(t)\,u(t)
\end{aligned}
$$

Part b)

Note that the integral in the unilateral Laplace transform will not catch the anti-causal term, $-e^{t}u(-t)$, so

$$
\begin{aligned}
\mathcal{X}(s) &= \int_{0^-}^{\infty} [-e^{t}u(-t) - e^{-t}\cos(t)\,u(t)]e^{-st}dt \\
&= \int_{0^-}^{\infty} -e^{-t}\cos(t)\,u(t)e^{-st}dt
\end{aligned}
$$

Next, since the causal term $-e^{-t}\cos(t)\,u(t) = 0\ \forall t < 0$, we can extend the bound of the integral to $-\infty$.

$$
\mathcal{X}(s) = \int_{-\infty}^{\infty} -e^{-t}\cos(t)\,u(t)e^{-st}dt
$$

This is the bilateral Laplace transform of $-e^{-t}\cos(t)\,u(t)$, so

$$
\mathcal{X}(s) = -\frac{s+1}{(s+1)^2+1}
$$

Remark: Although $-e^{t=0}u\big(-(t=0)\big) = -1$, this term does not contribute to the integral as the area under a single finite-valued point is 0. (A more rigorous justification would involve Lebesgue integration)

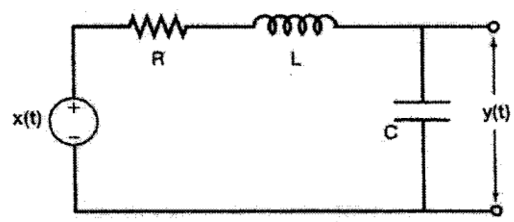
5. (20 points) Consider the RLC circuit below governed by the differential equation:

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t).$$

a) (6 points) Determine the transfer function of the LTI system implemented with this circuit.

b) (7 points) How should R, L and C be related so that there is no oscillation in the step response?

c) (7 points) How should R, L and C be related so that there is no resonance peak in the magnitude of the frequency response, $|H(j\omega)|$?



a) Taking Laplace transform, we obtain

$$LC\, s^2\, Y(s) + RC\, s\, Y(s) + Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{LC\,s^2 + RC\,s + 1}$$

b) $x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$

$$Y(s) = X(s) \cdot H(s) = \frac{1}{s\,(LC\,s^2 + RC\,s + 1)}$$

To avoid oscilations all poles must be real

Hence, $R^2C^2 - 4LC \geq 0$

$$\boxed{R \geq 2\sqrt{\frac{L}{C}}}$$

c) Second-order system from class:

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{\frac{1}{\omega_n^2} s^2 + 2 \cdot \frac{\xi}{\omega_n} s + 1}$$

Comparing this with $H(s)$, we get

$$\frac{1}{\omega_n^2} = LC \quad , \quad 2 \cdot \frac{\xi}{\omega_n} = RC \quad \Rightarrow \quad \omega_n = \frac{1}{\sqrt{LC}}$$

$$\xi = \frac{1}{2} R \sqrt{\frac{C}{L}}$$

There is no resonance peak if $\xi \geq \frac{1}{\sqrt{2}}$

$$\Leftrightarrow \frac{1}{2} R \sqrt{\frac{C}{L}} \geq \frac{1}{\sqrt{2}} \quad \Leftrightarrow \quad \boxed{R \geq \sqrt{\frac{2L}{C}}}$$