**University of California, Berkeley**
**Department of Mechanical Engineering**
**ME 104, Fall 2013**

**Midterm Exam 2 Solutions**

**1**. (30 points) Consider a particle $X$, of mass $m$, moving along a plane curve $\mathcal{C}$, which is parametrized by its arc length $s$. The unit tangent vector $\mathbf{e}_t$ to $\mathcal{C}$ is defined by

$$\mathbf{e}_t = \frac{d\mathbf{r}}{ds}, \tag{1}$$

and the principal unit normal vector to $\mathcal{C}$ is

$$\mathbf{e}_n = \rho\,\frac{d\mathbf{e}_t}{ds}, \tag{2}$$

where $\rho$ ($> 0$) is the radius of curvature of $\mathcal{C}$.

**(a)** Show that the velocity and acceleration vectors of $X$ may be expressed as

$$\mathbf{v} = \dot{s}\,\mathbf{e}_t = v\,\mathbf{e}_t, \qquad \mathbf{a} = \dot{v}\,\mathbf{e}_t + \frac{v^2}{\rho}\,\mathbf{e}_n. \tag{3}$$

**(Solution)** (10 points)

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{ds}\dot{s} = \dot{s}\,\mathbf{e}_t = v\,\mathbf{e}_t$$

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{v}\,\mathbf{e}_t + v\,\dot{\mathbf{e}}_t = \dot{v}\,\mathbf{e}_t + v\frac{d\mathbf{e}_t}{ds}\dot{s} = \dot{v}\,\mathbf{e}_t + \frac{v^2}{\rho}\,\mathbf{e}_n.$$

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**(b)** Suppose that a car weighing 3000 lbf is being tested on a circular skidpad of radius 100 ft. If the maximum sustainable constant speed of the car is 37 mph, calculate the acceleration of the car in $g$'s. Also, calculate the total frictional force that the asphalt surface then exerts on the tires.

**(Solution)** (10 points)

The car has a constant speed $v = 37$ mph, so

$$\mathbf{a} = 0\,\mathbf{e}_t + \frac{v^2}{\rho}\,\mathbf{e}_n = \frac{v^2}{\rho}\,\mathbf{e}_n.$$

The acceleration of the car is

$$a_n = \frac{v^2}{r} = \left(\frac{37 \times 5280}{3600}\right)^2 \left(\frac{1}{100}\right) = 29.4487 \text{ ft/s}^2 = 0.915\,g.$$

The total frictional force on the tires is

$$F = ma_n = m\frac{v^2}{r} = \left(\frac{3000}{32.2}\right) 29.4487 = 2740 \text{ lbf}.$$

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**(c)** Suppose that a projectile is launched with speed $v_0$ at an angle 45° from the horizontal plane. Neglect air resistance. Draw the free-body diagram of the projectile at the highest point $A$ of its trajectory. Find the velocity vector $\mathbf{v}(A)$ and acceleration vector $\mathbf{a}(A)$ of the projectile, and indicate them on your sketch. What are the normal and tangential components of $\mathbf{a}(A)$? Express the radius of curvature $\rho(A)$ in terms of $v_0$ and $g$.

**(Solution)** (10 points)

At the highest point $A$, the velocity $\mathbf{v}$ is given by

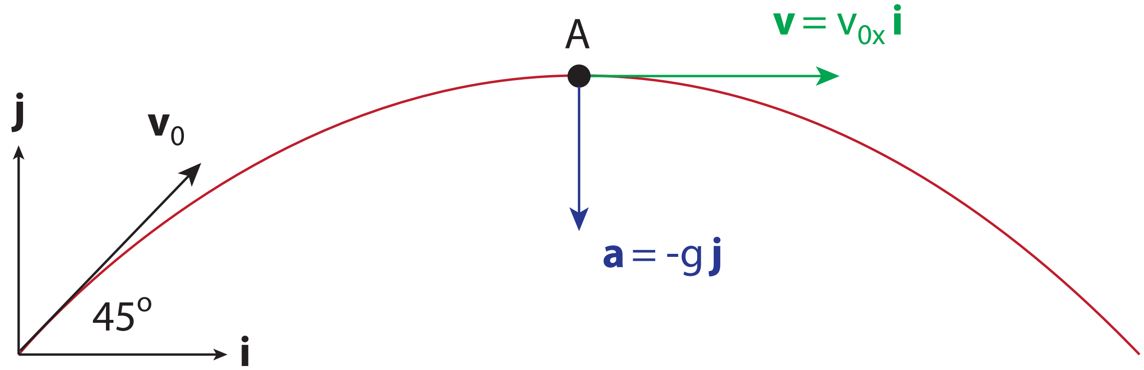
$$\mathbf{v}(A) = v_{0x}\,\mathbf{i} + 0\,\mathbf{j} = \frac{v_0}{\sqrt{2}}\,\mathbf{i}$$

The acceleration is

$$\mathbf{a}(A) = -g\,\mathbf{j} = g\,\mathbf{e}_n = \frac{v^2}{\rho}\,\mathbf{e}_n.$$

The tangential component of $\mathbf{a}$ is zero. Taking the normal component,

$$g = \frac{v_0^2}{2\rho} \qquad \longrightarrow \qquad \rho(A) = \frac{v_0^2}{2g}$$

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**2**. (35 points) Recall that the linear impulse of a force $\mathbf{F}$ is defined by

$$\mathcal{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt. \tag{4}$$

**(a)** If $\mathbf{G}$ is the linear momentum of a particle of mass $m$, prove that

$$\mathcal{I} = \Delta\mathbf{G} = \mathbf{G}_2 - \mathbf{G}_1. \tag{5}$$

**(Solution)** (5 points)

$$\mathcal{I} = \int_{t_1}^{t_2} \mathbf{F}\, dt = \int_{t_1}^{t_2} \dot{\mathbf{G}}\, dt = \int_{t_1}^{t_2} \frac{d\mathbf{G}}{dt}\, dt = \int_{\mathbf{G}_1}^{\mathbf{G}_2} d\mathbf{G} = \mathbf{G}_2 - \mathbf{G}_1 = \Delta\mathbf{G}.$$

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**(b)** A simplified model of a ballistic pendulum comprises: (i) a simple pendulum consisting of an inextensible string of length $\ell$ and a bob of mass $M$, which is initially at rest in its stable equilibrium position; and (ii) a bullet of mass $m$ travelling with speed $v$ on a collision course with the bob. Treat the collision as impulsive, and calculate the speed $v^+$ of the bob and the embedded bullet immediately after impact.

**(Solution)** (10 points)

Gravitation force is neglected during impact, and the impulsive forces are entirely internal to the system. Thus, the linear momentum is conserved:

$$\mathbf{G}_2 = \mathbf{G}_1 \qquad \rightarrow \qquad mv\,\mathbf{j} = (m+M)v^+\,\mathbf{j} \qquad \rightarrow \qquad v^+ = \frac{m}{m+M}v.$$

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**(c)** Show that the maximum angle $\theta_m$ reached by the pendulum after impact is given by

$$1 - \cos\theta_m = \frac{(v^+)^2}{2g\ell}. \tag{6}$$

**(Solution)** (10 points)

Using conservation of energy between the positions $\theta = \theta_m$ and $\theta = 0$, we have

$$T^+ + V^+ = 0 + V_m$$
$$\frac{1}{2}(M+m)(v^+)^2 - (M+m)g\ell = 0 - (M+m)g\ell\cos\theta_m,$$

or

$$\frac{1}{2}(M+m)(v^+)^2 = (M+m)g\ell(1-\cos\theta_m).$$

Rearrange to get

$$1 - \cos\theta_m = \frac{(v^+)^2}{2g\ell}.$$

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**(d)** If

$$\ell = 5 \text{ m}, \qquad m = 0.2 \text{ kg}, \qquad M = 20 \text{ kg}, \qquad \theta_m = 10^\circ,$$

calculate the incoming speed $v$ of the bullet, and express it as a Mach number, using $c = 343$ m/s for the speed of sound in air at a temperature of 20°C.

**(Solution)** (10 points)

From Parts **(b)** and **(c)**,

$$\left(\frac{m}{m+M}\right)^2 v^2 = 2g\ell(1 - \cos\theta_m).$$

Solve for the speed $v$ and substitute the given values:

$$\begin{aligned} v &= \left(\frac{m+M}{m}\right)\sqrt{2g\ell(1-\cos\theta_m)} \\ &= \left(\frac{0.2+20}{0.2}\right)\sqrt{2 \times 9.81 \times 5 \times (1 - \cos 10^\circ)} \\ &= 123.3 \text{ m/s} \qquad \longrightarrow \qquad Ma = \frac{v}{c} = \frac{123.3}{343} = 0.36. \end{aligned}$$

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**3**. (35 points) Consider a planet of mass $m$ orbiting the sun (mass $M$), regarded as fixed in an inertial frame of reference. Use polar coordinates, and recall that the velocity of the planet is given by

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta. \tag{7}$$

**(a)** Calculate the angular momentum $\mathbf{H}_O$ of the planet about the center of the sun, and deduce that

$$r^2\dot{\theta} = const. = h. \tag{8}$$

**(Solution)** (8 points)

The position vector of the planet is $\mathbf{r} = r\,\mathbf{e}_r$. The angular momentum

$$\begin{aligned} \mathbf{H}_O &= \mathbf{r} \times m\mathbf{v} \\ &= r\,\mathbf{e}_r \times m\left(\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta\right) \\ &= mr^2\dot{\theta}\,\mathbf{k}. \end{aligned}$$

The moment $\mathbf{M}_O$ about the center of the sun is zero, which means $\dot{\mathbf{H}}_O = \mathbf{0}$. Thus, $\mathbf{H}_O$ is a constant vector. Then

$$\frac{\|\mathbf{H}_O\|}{m} = r^2\dot{\theta} = const. = h.$$

This constant, $h$, is the specific angular momentum of the planet.

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**(b)** Recall that the gravitational potential energy of the planet is $V = -\dfrac{GMm}{r}$, where $G = 6.6734 \times 10^{-11}$ m$^3$/kg.s$^2$, and the mass of the sun is $1.990 \times 10^{30}$ kg. Deduce that

$$\frac{1}{2}v^2 - \frac{GM}{r} = const. = \frac{E}{m}. \tag{9}$$

**(Solution)** (7 points)

The Newtonian gravitational force is a conservative force field. Hence, energy is conserved.

$$T + V = \frac{1}{2}mv^2 - \frac{GMm}{r} = E = const.$$

Divide this equation by the mass $m$ to get (9).

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**(c)** For the planet Mercury, the distances to perihelion and aphelion are

$$r_p = 46 \times 10^6 \text{ km}, \qquad r_a = 69.82 \times 10^6 \text{ km}. \tag{10}$$

If the speed of Mercury at aphelion is $38.86 \times 10^3$ m/s, calculate its speed at perihelion.

**(Solution)** (7 points)

The constant specific angular momentum

$$h = r_a v_a = r_p v_p.$$

Thus,

$$v_p = v_a \frac{r_a}{r_p} = \left(38.86 \times 10^3 \text{ m/s}\right) \frac{69.82 \times 10^6 \text{ km}}{46 \times 10^6 \text{ km}} = 58.98 \times 10^3 \text{ m/s}.$$

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**(d)** Using aphelion data, calculate the value of specific orbital energy of Mercury.

**(Solution)** (7 points)

Directly using Eq. (9), the specific orbital energy

$$\begin{aligned} \frac{E}{m} &= \frac{1}{2}v_a^2 - \frac{GM}{r_a} \\ &= \frac{1}{2}(38.86 \times 10^3 \text{ m/s})^2 - \frac{(6.6734 \times 10^{-11} \text{ m}^3/\text{kg.s}^2)(1.990 \times 10^{30} \text{ kg})}{69.82 \times 10^9 \text{ m}} \\ &= -1.147 \times 10^9 \text{ m}^2/\text{s}^2 \end{aligned}$$

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**(e)** Where in its orbit does the maximum acceleration of Mercury occur? Calculate its value.

**(Solution)** (6 points)

Recall the equation of orbital motion $\mathbf{F} = m\mathbf{a}$:

$$-\frac{GMm}{r^2}\,\mathbf{e}_r = m(\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\mathbf{e}_\theta$$

The magnitude of the acceleration $a = \|\mathbf{F}\|/m = GM/r^2$. It's clear that the maximum acceleration occurs at minimum $r$, which is at perihelion. This maximum value is

$$\begin{aligned} a_{max} = \frac{GM}{r_p^2} &= \frac{(6.6734 \times 10^{-11} \text{ m}^3/\text{kg.s}^2)(1.990 \times 10^{30} \text{ kg})}{(46 \times 10^9 \text{ m})^2} \\ &= 6.276 \times 10^{-2} \text{ m/s}^2. \end{aligned}$$

Note that

$$a_{max} \neq r_p\dot{\theta}_p^2 = \frac{v_p^2}{r_p} = 7.563 \times 10^{-2} \text{ m/s}^2$$

because $\ddot{r}_p$ is not zero, and $r_p \neq \rho = 55.4 \times 10^9$ m.

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