**University of California, Berkeley**
**Department of Mechanical Engineering**
**ME 104, Fall 2013**

**Midterm Exam 1 Solutions**

**1**. (20 points) **(a)** For a particle undergoing a rectilinear motion, the position, velocity, and acceleration vectors are given by

$$\mathbf{r} = x\,\mathbf{i}, \qquad \mathbf{v} = \dot{x}\,\mathbf{i} = v\,\mathbf{i}, \qquad \mathbf{a} = \ddot{x}\,\mathbf{i} = a\,\mathbf{i}. \tag{1}$$

Show that

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right). \tag{2}$$

**(Solution)** (5 points)

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}(2v)\frac{dv}{dx} = v\frac{dv}{dx} = \frac{dx}{dt}\frac{dv}{dx} = \frac{dv}{dt} = a$$

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**(b)** Consider a particle $P$ of mass $m$ travelling in a horizontal plane. Let its velocity vector be

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} = v_x\,\mathbf{i} + v_y\,\mathbf{j}. \tag{3}$$

Suppose that $v_x^2$ and $v_y^2$ are specified by

$$v_x^2 = C_1 - \frac{1}{2}kx^2 \geq 0, \qquad v_y^2 = C_2 - \frac{1}{2}ky^2 \geq 0, \tag{4}$$

where $k$ is a positive constant, and $C_1$ and $C_2$ are constants. Calculate $a_x$ and $a_y$.

**(Solution)** (5 points)

Using (2),

$$a_x = \frac{d}{dx}\left[\frac{1}{2}\left(C_1 - \frac{1}{2}kx^2\right)\right] = -\frac{kx}{2}, \qquad a_y = \frac{d}{dy}\left[\frac{1}{2}\left(C_2 - \frac{1}{2}ky^2\right)\right] = -\frac{ky}{2}$$

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**(c)** Show that the force acting on $P$ is centripetal.

**(Solution)** (5 points)

The force acting on $P$ can be obtained using Newton's second law:

$$\mathbf{F} = m\mathbf{a}$$
$$F_x\,\mathbf{i} + F_y\,\mathbf{j} = ma_x\,\mathbf{i} + ma_y\,\mathbf{j}.$$

Taking the inner product with **i** and **j** respectively, and using the results from Part **(b)**,

$$F_x = ma_x = -\frac{mkx}{2}, \qquad F_y = ma_y = -\frac{mky}{2}.$$

Then,

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} = -\frac{mk}{2}\,(x\,\mathbf{i} + y\,\mathbf{j}) = -\frac{mk}{2}\,\mathbf{e}_r.$$

The direction of the force is $-\mathbf{e}_r$. The force acting on $P$ is centripetal.

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**(d)** Obtain the differential equations for the $x$ and $y$ coordinates of $P$, and provide a simple solution of either one of them.

**(Solution)** (5 points)

From the acceleration components $a_x$ and $a_y$, we have

$$\ddot{x} + \frac{k}{2}x = 0, \qquad \ddot{y} + \frac{k}{2}y = 0.$$

Both equations represent simple harmonic motion. We can suppose that the solution has the form

$$x = A\cos(\omega t) + B\sin(\omega t).$$

Differentiate once to get

$$\dot{x} = -\omega A\sin(\omega t) + \omega B\cos(\omega t).$$

Differentiate again to get

$$\ddot{x} = -\omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right].$$

Substituting into the differential equation,

$$\begin{aligned} -\omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right] + \frac{k}{2}\left[A\cos(\omega t) + B\sin(\omega t)\right] &= 0 \\ \left[A\cos(\omega t) + B\sin(\omega t)\right]\left(-\omega^2 + \frac{k}{2}\right) &= 0 \end{aligned}$$

So $\omega = \sqrt{k/2}$. The solution becomes

$$x = A\cos\left(\sqrt{\frac{k}{2}}t\right) + B\sin\left(\sqrt{\frac{k}{2}}t\right)$$

We can choose the initial conditions to solve for $A$ and $B$ and complete the solution.

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**2**. (30 points) Let $OABC$ be a rigid plate which is rotating at constant angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k}$ around a vertical axis $OZ$ ($\omega = \dot{\theta} = const.$). Introduce a corotational basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ on the plate, as indicated in Fig. 1. Suppose that a rigid rod $OAD$ is welded to the plate and that $MN$ is a rigid guide ($OMN$ is a right angle.). Let $P$ be a pin of mass $m$ that can slide on the rod and inside the guide.

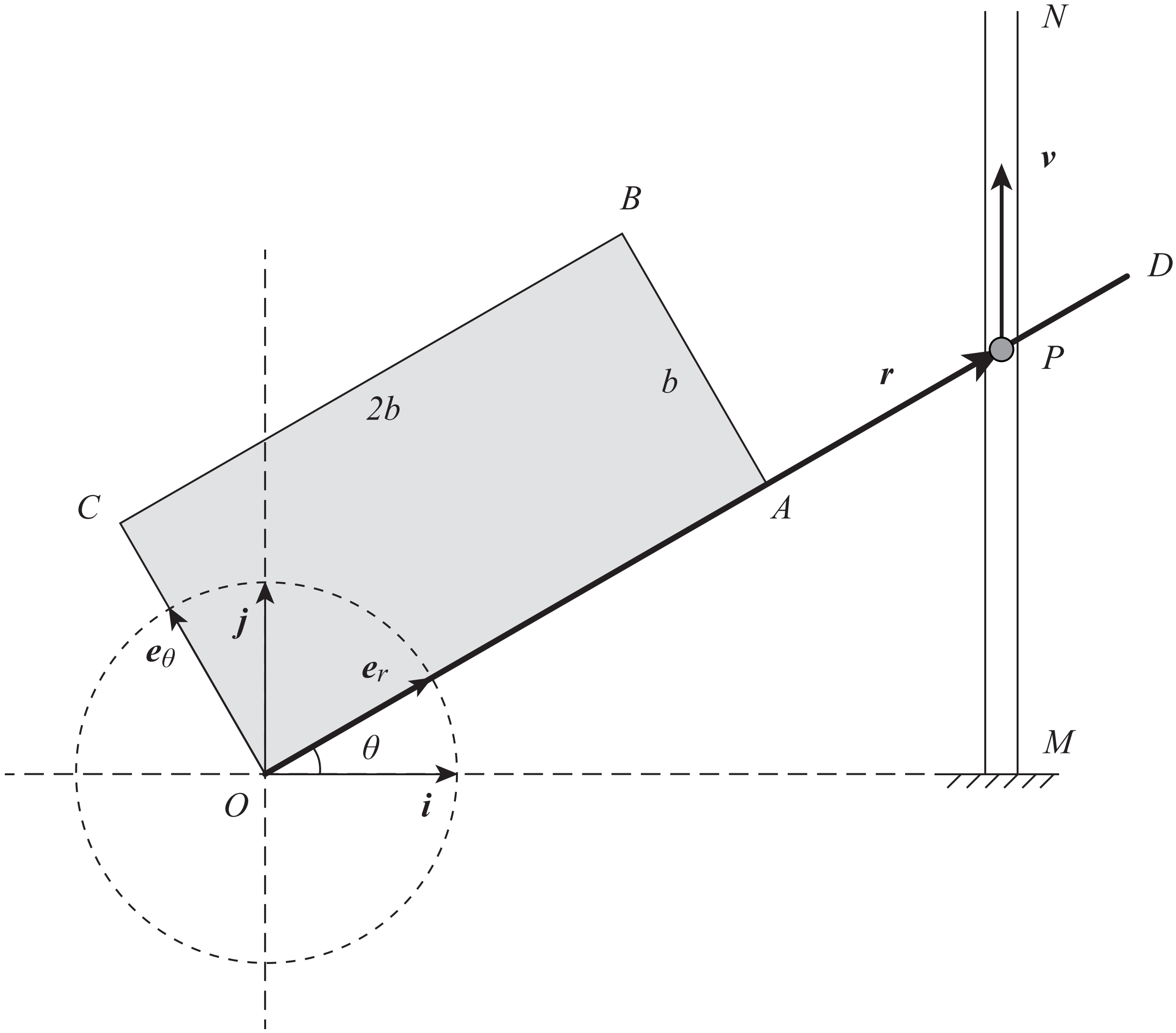


Figure 1: Problem 2

Figure 1.

**(a)** Write down the relationship between the corotational basis vectors $\mathbf{e}_r, \mathbf{e}_\theta$ and the fixed basis vectors $\mathbf{i}, \mathbf{j}$.

**(Solution)** (4 points)

This can either be written in matrix form:

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix},$$

or

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}. \end{aligned}$$

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**(b)** Show that

$$\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta = \boldsymbol{\omega} \times \mathbf{e}_r, \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\,\mathbf{e}_r = \boldsymbol{\omega} \times \mathbf{e}_\theta. \tag{5}$$

**(Solution)** (5 points)

Differentiating the cylindrical basis with respect to $t$,

$$\dot{\mathbf{e}}_r = \frac{d\mathbf{e}_r}{dt} = \dot{\theta}\,(-\sin\theta)\;\mathbf{i} + \dot{\theta}\cos\theta\,\mathbf{j} = \dot{\theta}\,\mathbf{e}_\theta,$$

$$\dot{\mathbf{e}}_\theta = \frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\cos\theta\,\mathbf{i} + \dot{\theta}\,(-\sin\theta)\;\mathbf{j} = -\dot{\theta}\,\mathbf{e}_r$$

Also,

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{e}_r &= \dot{\theta}\,\mathbf{k} \times \mathbf{e}_r = \dot{\theta}\,\mathbf{e}_\theta \\ \boldsymbol{\omega} \times \mathbf{e}_\theta &= \dot{\theta}\,\mathbf{k} \times \mathbf{e}_\theta = -\dot{\theta}\,\mathbf{e}_r \end{aligned}$$

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**(c)** Express the velocity and acceleration of $P$ on both bases, given that $\dot{\theta} = const$. Hence deduce that

$$\dot{r} = \dot{y}\,\mathbf{j} \cdot \mathbf{e}_r, \qquad r\dot{\theta} = \dot{y}\,\mathbf{j} \cdot \mathbf{e}_\theta. \tag{6}$$

**(Solution)** (5 points)

We note that in the motion of the pin, $v_x = \dot{x} = 0$. If $\mathbf{r} = r(t)\,\mathbf{e}_r(t) = x\,\mathbf{i} + y\,\mathbf{j}$, and with $\dot{\theta} = const$,

$$\begin{aligned} \mathbf{v} = \dot{\mathbf{r}} &= \dot{y}\,\mathbf{j} = v_y\,\mathbf{j} \\ &= \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r \\ &= \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta \\ &= v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta \end{aligned}$$

$$
\begin{aligned}
\mathbf{a} = \dot{\mathbf{v}} = \ddot{y}\,\mathbf{j} = a_y\,\mathbf{j} \\
&= \ddot{r}\,\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\dot{\theta}\mathbf{e}_\theta \\
&= \ddot{r}\,\mathbf{e}_r + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + \dot{r}\dot{\theta}\,\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r \\
&= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (2\dot{r}\dot{\theta})\,\mathbf{e}_\theta \\
&= a_r\,\mathbf{e}_r + a_\theta\,\mathbf{e}_\theta.
\end{aligned}
$$

We can take the inner product of the velocity with $\mathbf{e}_r$ and $\mathbf{e}_\theta$ to get the equations in (6):

$$
\mathbf{v}\cdot\mathbf{e}_r = \dot{r} = \dot{y}\,\mathbf{j}\cdot\mathbf{e}_r, \qquad \mathbf{v}\cdot\mathbf{e}_\theta = r\dot{\theta} = \dot{y}\,\mathbf{j}\cdot\mathbf{e}_\theta
$$

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**(d)** Suppose that at $\theta = 30°$,

$$
r = 0.05\ \text{m}, \qquad \dot{r} = 0.2\ \text{m/s}, \qquad \ddot{r} = -0.025\ \text{m/s}^2. \tag{7}
$$

(i) Calculate $\dot{\theta}$ and $\dot{y}$, and check that $\dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$.
(ii) Find $\ddot{y}$.
(iii) Calculate the magnitude of the total force $\mathbf{F}$ acting on $P$ if the mass of $P$ is $m = 0.5$ kg.

**(Solution)** (7 points)

The results from parts (a) and (c) gives us the component equations

$$
\dot{r} = \dot{y}\,\mathbf{j}\cdot\mathbf{e}_r = \dot{y}\sin\theta, \qquad \text{and} \qquad r\dot{\theta} = \dot{y}\,\mathbf{j}\cdot\mathbf{e}_\theta = \dot{y}\cos\theta.
$$

(i) Dividing the equations,

$$
\begin{aligned}
\frac{\dot{r}}{r\dot{\theta}} = \tan\theta \qquad \longrightarrow \qquad \dot{\theta} &= \frac{\dot{r}}{r\tan\theta} \\
&= \frac{0.2\ \text{m/s}}{0.05\ \text{m}\times\tan 30°} \\
&= \frac{1/5}{1/20\times 1/\sqrt{3}}\ \text{rad/s} \\
&= 4\sqrt{3}\ \text{rad/s} \\
&= 6.93\ \text{rad/s}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\dot{r} = \dot{y}\sin\theta \qquad \longrightarrow \qquad \dot{y} &= \frac{\dot{r}}{\sin\theta} \\
&= \frac{0.2\ \text{m/s}}{\sin 30°} \\
&= 0.4\ \text{m/s}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{y}^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\
(0.4\ \mathrm{m/s})^2 &= (0.2\ \mathrm{m/s})^2 + (0.05\ \mathrm{m})^2 \times \left(4\sqrt{3}\ \mathrm{rad/s}\right)^2 \\
0.16\ \mathrm{m^2/s^2} &= 0.04\ \mathrm{m^2/s^2} + 0.12\ \mathrm{m^2/s^2} \\
&= 0.16\ \mathrm{m^2/s^2}. \quad \surd
\end{aligned}
$$

(ii) You are correct if you used the following methods to find $\ddot{y}$.

$$
\begin{aligned}
& (1) && \ddot{y}\,\mathbf{j}\cdot\mathbf{j} = \mathbf{a}\cdot\mathbf{j} = (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r\cdot\mathbf{j} + (2\dot{r}\dot{\theta})\,\mathbf{e}_\theta\cdot\mathbf{j} \\
\text{or}\quad & (2) && \ddot{y}\,\mathbf{j}\cdot\mathbf{e}_r = \mathbf{a}\cdot\mathbf{e}_r = \ddot{r} - r\dot{\theta}^2 \\
\text{or}\quad & (3) && \ddot{y}\,\mathbf{j}\cdot\mathbf{e}_\theta = \mathbf{a}\cdot\mathbf{e}_\theta = 2\dot{r}\dot{\theta}
\end{aligned}
$$

It is given in the problem that $\dot{\theta} = const$. With the introduction of a value for $\ddot{r}$, the problem actually becomes over-determined. This means a value is specified which can actually be calculated, and if we don't choose to specify it correctly, we have a contradiction. As a result, $\ddot{y}$ does not have a unique solution. For the sake of completeness, the actual values of $\ddot{r}$ and $\ddot{y}$ are solved below.

$$
\begin{aligned}
\ddot{y}\,\mathbf{j}\cdot\mathbf{e}_\theta = 2\dot{r}\dot{\theta} \quad &\rightarrow \quad \ddot{y} = \frac{2\dot{r}\dot{\theta}}{\cos\theta} = \frac{2\times 0.2\times 4\sqrt{3}}{\cos 30} = 3.2\ \mathrm{m/s^2} \\
\ddot{y}\,\mathbf{j}\cdot\mathbf{e}_r = \ddot{r} - r\dot{\theta}^2 \quad &\rightarrow \quad \ddot{r} = \ddot{y}\sin\theta + r\dot{\theta}^2 = 3.2\sin 30 + 0.05\times(4\sqrt{3})^2 = 4\ \mathrm{m/s^2}
\end{aligned}
$$

(iii) The magnitude of the force is

$$
\begin{aligned}
\|\mathbf{F}\| = \|m\mathbf{a}\| = m\|\mathbf{a}\| = m\|\ddot{y}\,\mathbf{j}\| &= m\ddot{y} \\
&= 0.5\ \mathrm{kg}\times 3.2\ \mathrm{m/s^2} \\
&= 1.6\ \mathrm{N}.
\end{aligned}
$$

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**(e)** Let $\eta$ represent the distance along the diagonal $OB$ of the plate. Calculate the velocity distribution $\mathbf{v}(\eta)$ along $OB$ and sketch it.

**(Solution)** (5 points)

We can set up a new basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ by rotating the basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ at an angle $\phi = \tan^{-1}(1/2) = 26.565°$ such that $\mathbf{e}_1$ is along the diagonal of the plate and $\mathbf{e}_2$ is perpendicular to it. See Figure 2. The unit vector $\mathbf{e}_1$ along the diagonal of the rigid plate is

$$
\mathbf{e}_1 = \cos\phi\,\mathbf{e}_r + \sin\phi\,\mathbf{e}_\theta = \frac{1}{\sqrt{5}}\left(2\,\mathbf{e}_r + \mathbf{e}_\theta\right).
$$

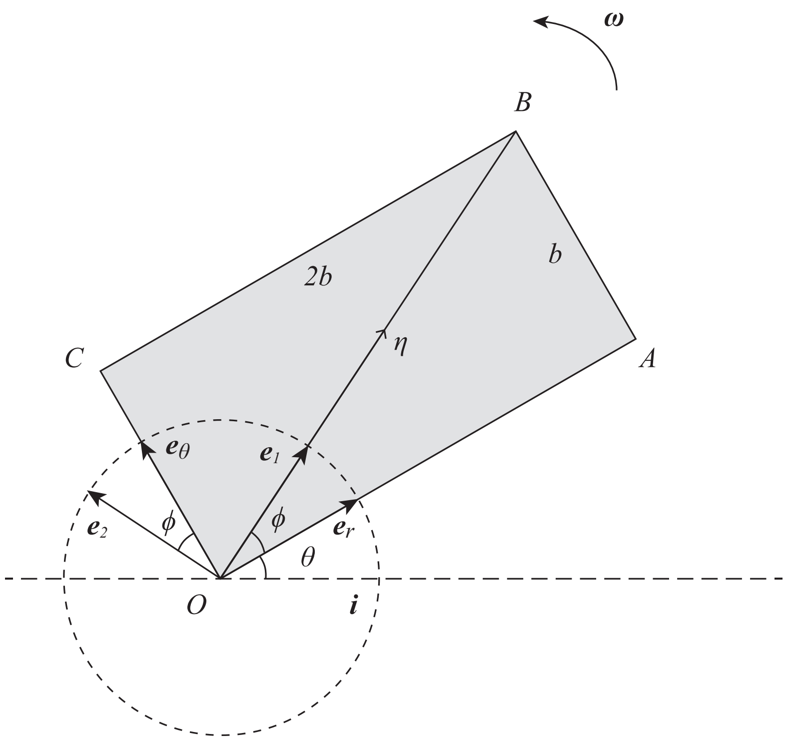


Figure 2: Rotating plate

The unit vector perpendicular to the diagonal is

$$\mathbf{e}_2 = -\sin\phi\,\mathbf{e}_r + \cos\phi\,\mathbf{e}_\theta = \frac{1}{\sqrt{5}}\left(-\mathbf{e}_r + 2\,\mathbf{e}_\theta\right).$$

Also, $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{k}$. Their time derivatives are

$$\dot{\mathbf{e}}_1 = \boldsymbol{\omega} \times \mathbf{e}_1 = \dot{\theta}\,\mathbf{e}_2, \qquad \dot{\mathbf{e}}_2 = \boldsymbol{\omega} \times \mathbf{e}_2 = -\dot{\theta}\,\mathbf{e}_1.$$

Along $OB$,

$$\begin{aligned}\mathbf{r}(\eta) = \eta\,\mathbf{e}_1 &= \eta\left(\cos\phi\,\mathbf{e}_r + \sin\phi\,\mathbf{e}_\theta\right) \\ &= \frac{\eta}{\sqrt{5}}\left(2\,\mathbf{e}_r + \mathbf{e}_\theta\right).\end{aligned}$$

The velocity is

$$\begin{aligned}\mathbf{v} = \dot{\mathbf{r}} &= \eta\,\dot{\mathbf{e}}_1 = \eta\,\boldsymbol{\omega} \times \mathbf{e}_1 \\ &= \boldsymbol{\omega} \times \mathbf{r} = \dot{\theta}\,\mathbf{k} \times \eta\,\mathbf{e}_1 = \dot{\theta}\eta\,\mathbf{k} \times \mathbf{e}_1 \\ &= \eta\,\dot{\theta}\,\mathbf{e}_2 \\ &= \eta\,\dot{\theta}\,\left(-\sin\phi\,\mathbf{e}_r + \cos\phi\,\mathbf{e}_\theta\right).\end{aligned}$$

A sketch of the velocity is shown in Figure 3.

**(f)** Also calculate the acceleration along the diagonal $OB$ and indicate its variation on a sketch ($\dot{\theta} = const.$).

**(Solution)** (4 points)

The acceleration is

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} &= \eta\,\dot{\theta}\,\dot{\mathbf{e}}_2 = \eta\,\dot{\theta}\,\boldsymbol{\omega} \times \mathbf{e}_2 \\ &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \dot{\theta}\,\mathbf{k} \times (\eta\,\dot{\theta}\,\mathbf{k} \times \mathbf{e}_1) \\ &= -\eta\,\dot{\theta}^2\,\mathbf{e}_1 \\ &= -\dot{\theta}^2\,\mathbf{r}(\eta),\end{aligned}$$

which is centripetal. A sketch is shown in Figure 3.

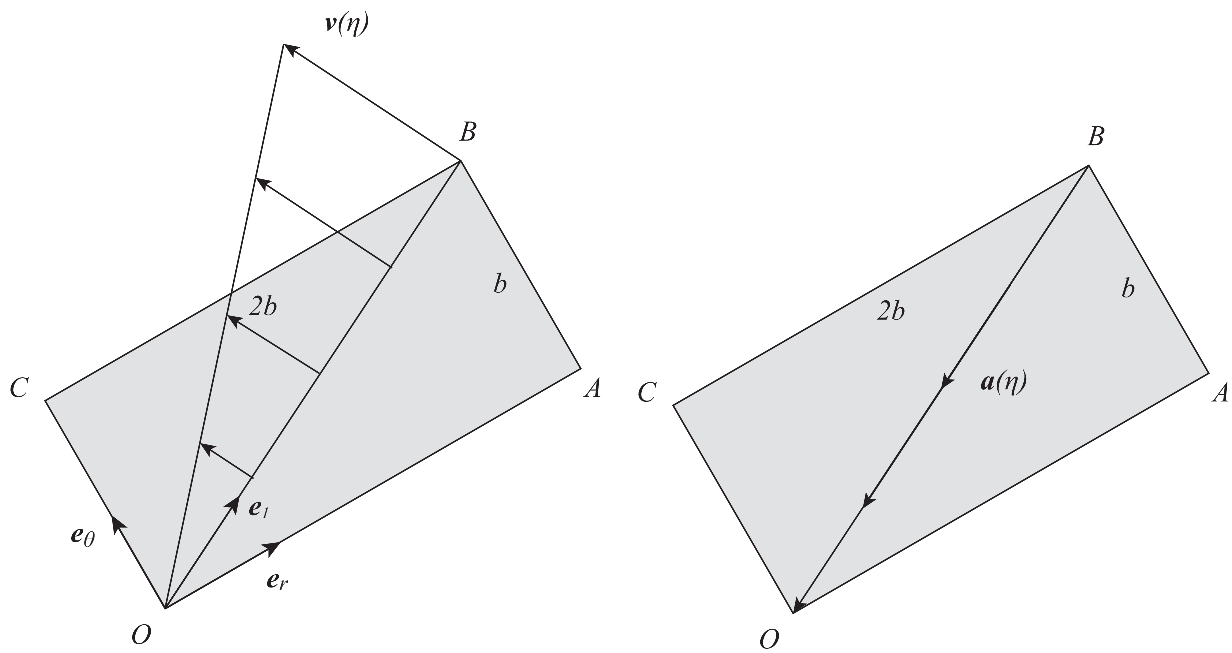


Figure 3: Velocity and acceleration