Solutions

Q1: Warm-up

a)

First we determine the DTFT for each signal:

$$\frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}e^{-i\omega} + \frac{1}{2}e^{i\omega} = \cos(\omega)$$
$$\frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+1] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}e^{-i\omega} - \frac{1}{2}e^{i\omega} = -\sin(\omega)$$
$$\frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}e^{i\omega} - \frac{1}{2}e^{-i\omega} = \sin(\omega)$$

Now, we want to take the magnitude of the DTFTs. We know the function should have zeros at values of $[-\pi, 0, \pi]$. This only holds for the *sin* terms. Therefore, the second and third bubbles should be selected.

Q2: I've Seen Better Phase

a) S is causal?

True. The definition of causality states that $y[n]$ must only depend on current or previous values of $x[n]$. We have the input to system T, which is $x[n]e^{i\frac{2\pi}{3}n}$. $x[n]e^{i\frac{2\pi}{3}n}$ is causal since it only depends on current values of $x[n]$. We also know that T is causal, and cascading causal systems results in a causal system. Therefore S is causal. A common misconception was that a causal system has $y[n] = 0$ or $x[n] = 0$ for $n < 0$. This is not true. Rather, we can say that in a causal system, $h[n] = 0$ for $n < 0$.

b) S is linear?

True. $x[n]e^{i\frac{2\pi}{3}n}$ is a linear function of $x[n]$. This is because $e^{i\frac{2\pi}{3}n}$ is not a function of $x[n]$. We can always resort to the test of replacing $x[n]$ with $\alpha_1 x_1[n] + \alpha_2 x_2[n]$. We get

$$y[n] = \alpha_1 x_1[n]e^{i\frac{2\pi}{3}n} + \alpha_2 x_2[n]e^{i\frac{2\pi}{3}n}$$
$$= \alpha_1 y_1[n] + \alpha_2 y_2[n],$$

thus the first part is linear. We also know that T is linear and cascading linear systems is linear.

c) S is time invariant?

False. $x[n]e^{i\frac{2\pi}{3}n}$ will behave differently at different times n. We can also use the test.

$$u[n] = x[n]e^{i\frac{2\pi}{3}n}$$
$$u[n-K] = x[n-K]e^{i\frac{2\pi}{3}(n-K)}$$

$$v[n] = x[n-K]$$
$$w[n] = v[n]e^{i\frac{2\pi}{3}n}$$
$$w[n] \neq u[n-K]$$

The system S can be time invariant if T undos the multiplication with $e^{i\frac{2\pi}{3}n}$. However, this would require T to be time variant, which is not the case. A common misconception was that a change in frequency implies TV. However, that is not always true. Here's a counter example: $y(t) = (x(t))^2$. Another misconception is that having a phase shift implies TV. A counter example for this is $y(t) = x(t-\tau)$.

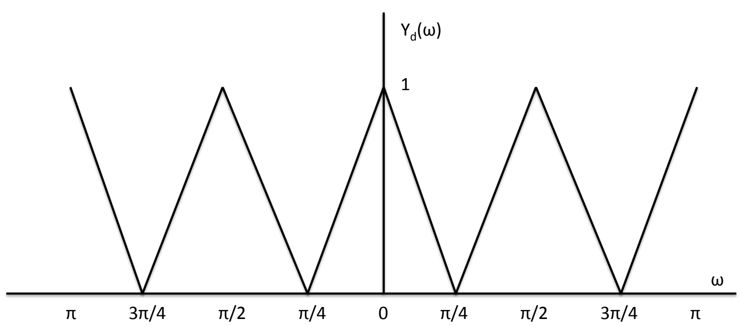
Q3: Proctor and Upsample

We have

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-i\omega n}.$$

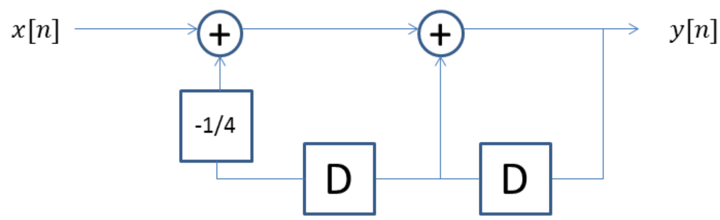
Since $y[n]$ is only nonzero for $n = 4m$ with integer m,

$$Y_d(\omega) = \sum_{m=-\infty}^{\infty} y[4m]e^{-i\omega 4m} = \sum_{m=-\infty}^{\infty} x[m]e^{-i4\omega m} = X_d(4\omega).$$



Q4: Bob the Filter

a) The block diagram is shown below, where D represents a unit delay.



b) Taking the DTFT of both sides of the equation, we get:

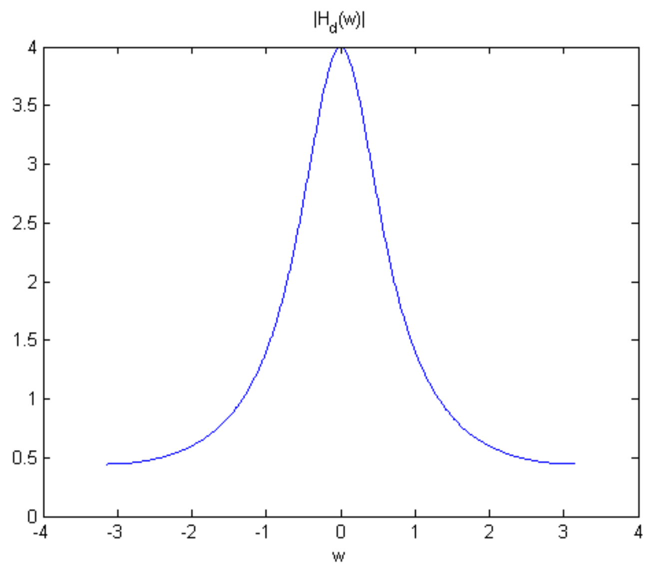
$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$$
$$Y_d(\omega) - Y_d(\omega)e^{-i\omega} + \frac{1}{4}Y_d(\omega)e^{-i\omega 2} = X_d(\omega)$$

And we can solve for the frequency response as:

$$\begin{aligned} H_d(\omega) &= \frac{Y_d(\omega)}{X_d(\omega)} \\ &= \frac{1}{1 - e^{-i\omega} + \frac{1}{4}e^{-i2\omega}} \\ &= \frac{1}{(1 - \frac{1}{2}e^{-i\omega})(1 - \frac{1}{2}e^{-i\omega})} \end{aligned}$$

So the magnitude response is given by:

$$|H_d(\omega)| = \frac{1}{|1 - \frac{1}{2}e^{-i\omega}|^2}$$



Since this filter emphasizes low frequencies and de-emphasizes high frequencies, it is a low pass filter.

c) Note that $x[n]$ can be represented as $x[n] = e^{i\pi n}$, a sinusoid of frequency π. We compute the frequency response at this frequency to be:

$$\begin{aligned} H_d(\pi) &= \frac{1}{(1+\frac{1}{2})(1+\frac{1}{2})} \\ &= \frac{4}{9} \end{aligned}$$

So $y[n]$ will simply be:

$$y[n] = H_d(\pi)e^{i\pi n} = \frac{4}{9}(-1)^n$$

Q5: The Matrix: Evaluations

The DFT equations are:

$$\text{Analysis: } X[k] = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ik\omega_0 n} \quad \text{Synthesis: } x(n) = \sum_{k=0}^{p-1} X[k]e^{ik\omega_0 n},\ \omega_0 = \frac{2\pi}{p}$$

The 4-point DFT of $\underline{x} = \{a, b, c, d\}$ is $\underline{X} = \{A, B, C, D\}$.

a)

Write down the 4-point IDFT matrix that maps $\underline{X}$ to $\underline{x}$. Use complex notation in rectangular coordinates of the form $(r + is)$ for the entries.

Answer: If $X = Wx$, then $x = W^{-1}X$, where $W^{-1} = \frac{1}{N}W^H$ (This is a special case for symmetric matrices like DFT matrices.)

$$W^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

b)

Suppose $A = 0$, $B = 0$,$C = 1$,$D = 0$. What are a, b, c, and d?

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.25 \\ 0.25 \\ -0.25 \end{bmatrix}$$

Q6: They Only Differ by a T

a)

We know:

$$X_d(\omega) = \sum_{n=0}^{7} x[n]e^{-i\omega n}$$

$$Y[k] = X_d(\omega)|_{\omega=\frac{2\pi}{8}k} = \sum_{n=0}^{7} x[n]e^{-i\frac{2\pi}{8}kn}$$

By definition of the DFT, we also know that:

$$Y[k] = \sum_{n=0}^{7} y[n]e^{-i\frac{2\pi}{8}kn} = \sum_{n=0}^{7} x[n]e^{-i\frac{2\pi}{8}kn}$$

Since the DFT and IDFT are invertible, we know they are one-to-one. Therefore, $y[n] = x[n]$.

b)

$$X_d(\omega) = \sum_{n=0}^{7} x[n]e^{-i\omega n}$$

$$Z[k] = X_d(\omega)|_{\omega=\frac{2\pi}{16}k} = \sum_{n=0}^{7} x[n]e^{-i\frac{2\pi}{16}kn}$$

By definition of DFT:

$$Z[k] = \sum_{n=0}^{15} z[n]e^{-i\frac{2\pi}{16}kn} = \sum_{n=0}^{7} x[n]e^{-i\frac{2\pi}{16}kn}$$

In order for these two summations to be equal, we know $z[n] = 0, n \in \{8, 9, ..., 15\}$. Therefore,

$$z[n] = \begin{cases} x[n] & \text{if } n \in \{0, ..., 7\} \\ 0 & \text{if } n \in \{8, ..., 15\} \end{cases}$$

Q7: Periodicity Makes the World Go Round

Using the definition of convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

We also know that $h[n] = 0 \forall n < 0, n > 2$, which gives us the following:

$$y[n] = \sum_{k=0}^{2} h[k]x[n-k]$$

Now expand this and plug in for n:

$$y[0] = \sum_{k=0}^{2} h[k]x[0-k] = h[0]x[0] + h[1]x[-1] + h[2]x[-2] = 1 \times 1 + 2 \times -1 + 3 \times 1 = 2$$

$$y[1] = \sum_{k=0}^{2} h[k]x[1-k] = h[0]x[1] + h[1]x[0] + h[2]x[-1] = 1 \times -1 + 2 \times 1 + 3 \times -1 = -2$$

$$y[2] = \sum_{k=0}^{2} h[k]x[2-k] = h[0]x[2] + h[1]x[1] + h[2]x[0] = 1 \times 1 + 2 \times -1 + 3 \times 1 = 2$$

This pattern will continue so it follows that $y[n] = \{2, -2\}$ with period $p = 2$.

Q8: Extreme Makover: Comb Edition

a)

The frequency response of H_3 is the product of H_1 and H_2:

$$H_3(\omega) = H_1(\omega)H_2(\omega) = \frac{1}{1-\alpha_1 e^{-i3\omega}} \times \frac{1}{1-\alpha_2 e^{-i6\omega}}$$

The salient features of the filter occur at multiples of $\frac{\pi}{3}$, so we will observe H_3 at those points to determine what the values of α_1 and α_2 are:

$$H_3(0) = \frac{1}{1-\alpha_1} \times \frac{1}{1-\alpha_2}$$
$$H_3(\frac{\pi}{3}) = \frac{1}{1+\alpha_1} \times \frac{1}{1-\alpha_2}$$

1. $\alpha_1 = 0.1, \alpha_2 = 0.1$
   $H_3(0) = \frac{1}{1-0.1} \times \frac{1}{1-0.1} \approx 1.2$
   $H_3(\frac{\pi}{3}) = \frac{1}{1+0.1} \times \frac{1}{1-0.1} \approx 1$
   This is plot (a).

2. $\alpha_1 = 0.1, \alpha_2 = 0.9$
   $H_3(0) = \frac{1}{1-0.1} \times \frac{1}{1-0.9} \approx 11$
   $H_3(\frac{\pi}{3}) = \frac{1}{1+0.1} \times \frac{1}{1-0.9} \approx 9$
   This is plot (c).

3. $\alpha_1 = 0.9, \alpha_2 = 0.1$
   $H_3(0) = \frac{1}{1-0.9} \times \frac{1}{1-0.1} \approx 11$
   $H_3(\frac{\pi}{3}) = \frac{1}{1+0.9} \times \frac{1}{1-0.1} \approx 0.5$
   This is plot (b).

4. $\alpha_1 = 0.9, \alpha_2 = 0.9$
   $H_3(0) = \frac{1}{1-0.9} \times \frac{1}{1-0.9} \approx 100$
   $H_3(\frac{\pi}{3}) = \frac{1}{1+0.9} \times \frac{1}{1-0.9} \approx 5$
   This is plot (d).

So,
a) I
b) III
c) II
d) IV

b)

Recall that $\Omega_d = \frac{2\pi f_c}{f_s}$ where f_c is the frequency in CT, f_s is the sampling rate, and Ω_d is the corresponding DT frequency. Since the comb filter is a DT filter, we need to find the sampled signal before determining the output of the filter.

$$x[n] = cos(\frac{\pi}{3}n) + e^{-i\frac{5\pi}{6}n} + sin(\frac{2\pi}{3}n) + 16e^{-i0n}$$

Now the output of the filter follows from the graph of the frequency response (the filter is symmetric so cos and sin can be scaled by the frequency response as well, but this is not true in general!!):

$$y[n] = H_3(\frac{\pi}{3})cos(\frac{\pi}{3}n) + H_3(\frac{-5\pi}{6})e^{-i\frac{5\pi}{6}n} + H_3(\frac{2\pi}{3})sin(\frac{2\pi}{3}n) + H_3(0)16e^{-i0n}$$

$$\approx 5cos(\frac{\pi}{3}n) + 100sin(\frac{2\pi}{3}n) + 1600 \approx 100sin(\frac{2\pi}{3}n) + 1600$$

We also accepted $y(t)$ after "unsampling" $y[n]$ back to CT:

$$y(t) \approx 5cos(8000\pi t) + 100sin(16000\pi t) + 1600 \approx 100sin(16000\pi t) + 1600$$

Note that $H_3(\omega)$ has a phase, but we chose coefficients carefully so that either the magnitude or phase at these points is 0.