1. Problem 1

   (a) Assume $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$. Then

   $$\begin{aligned} y(t) &= x(t^2) \\ &= \alpha_1 x_1(t^2) + \alpha_2 x_2(t^2) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

   So system is linear. Now assume $\hat{x}(t) = x(t - \tau)$. Then

   $$\begin{aligned} \hat{y}(t) &= \hat{x}(t^2) = x(t^2 - \tau) \\ &\neq y(t - \tau) = x((t - \tau)^2) \end{aligned}$$

   So system is time-varying.

   (b) Assume $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$. Then

   $$\begin{aligned} y(t) &= x(3t) \\ &= \alpha_1 x_1(3t) + \alpha_2 x_2(3t) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

   So system is linear. Now assume $\hat{x}(t) = x(t - \tau)$. Then

   $$\begin{aligned} \hat{y}(t) &= \hat{x}(3t) = x(3t - \tau) \\ &\neq y(t - \tau) = x(3(t - \tau)) = x(3t - 3\tau) \end{aligned}$$

   So system is time-varying.

   (c) Assume $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$. Then

   $$\begin{aligned} y(t) &= |x(t)| \\ &= |\alpha_1 x_1(t) + \alpha_2 x_2(t)| \\ &\neq \alpha_1 |x_1(t)| + \alpha_2 |x_2(t)| \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

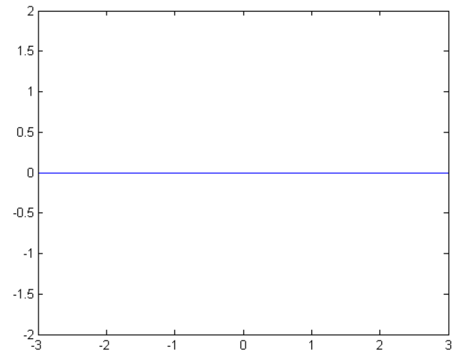
   So system is non-linear. Now assume $\hat{x}(t) = x(t - \tau)$. Then

   $$\begin{aligned} \hat{y}(t) &= |\hat{x}(t)| \\ &= |x(t - \tau)| = y(t - \tau) \end{aligned}$$

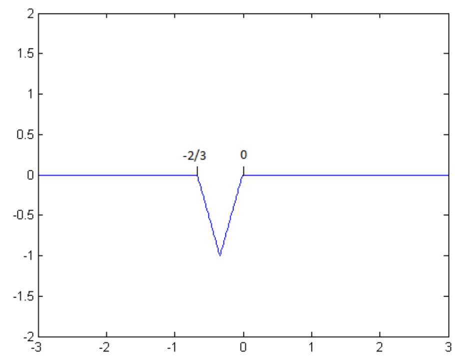
   So system is time-invariant.

2. Problem 2

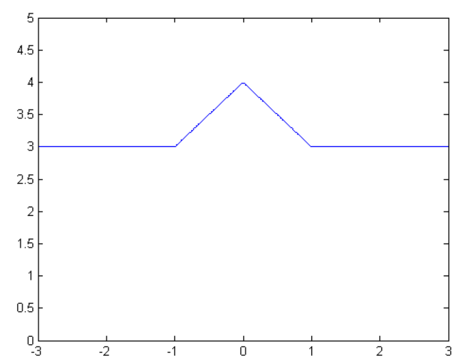
(a) Zero function



(b) Modified triangle function. Edges of triangle at $-2/3$ and 0.



(c) Modified triangle function. Edges of triangle at -1 and 1. Bias offset of 3.



3. Problem 3

(a) There isn't enough information to make a conclusion

Neither input-output pair show behavior that would not be true for an LTI system (e.g. no new frequency is introduced). However, two pairs of input-output is not enough to conclude that a system is LTI. For a system to be LTI, it must be true that for all possible input-output it exhibits LTI system behavior, but we do not know what the system will output for inputs other than the two given.

(b) The system is not LTI.

LTI system does not create new frequencies in the output that are not present in the input. Since the output has different frequencies than the input, the system cannot be LTI.

4. Problem 4

(a)

$$A_0 = \frac{1}{T}\int_{<T>} x(t)dt = \frac{1}{4}\int_{-1}^{1} 1dt = \frac{1}{2}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

(b) Since $x(t)$ is an even function, it will only have the cosine terms in its Fourier series summation (since sines are odd), so all B_k's are 0.

(c)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} B_k\sin(k\omega_0 t)$$

$$\begin{aligned}
y(t) &= 2x(t-1) \\
&= 2A_0 + 2\sum_{k=1}^{\infty} A_k\cos(k\omega_0(t-1)) + 2\sum_{k=1}^{\infty} B_k\sin(k\omega_0(t-1)) \\
&= 2A_0 + 2\sum_{k=1}^{\infty} A_k\cos(k\omega_0(t-1)) \qquad\qquad \text{(Since all } B_k = 0) \\
&= 2A_0 + \sum_{k=1}^{\infty} 2A_k\left(\cos(k\omega_0 t)\cos(k\omega_0) + \sin(k\omega_0 t)\sin(k\omega_0)\right) \\
&= 2A_0 + \sum_{k=1}^{\infty} 2A_k\cos(k\omega_0)\cos(k\omega_0 t) + \sum_{k=1}^{\infty} 2A_k\sin(k\omega_0)\sin(k\omega_0 t) \\
&= C_0 + \sum_{k=1}^{\infty} C_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} D_k\sin(k\omega_0 t)
\end{aligned}$$

$\rightarrow C_0 = 2A_0,\ C_k = 2A_k\cos(k\omega_0),\ D_k = 2A_k\sin(k\omega_0)$

(d)

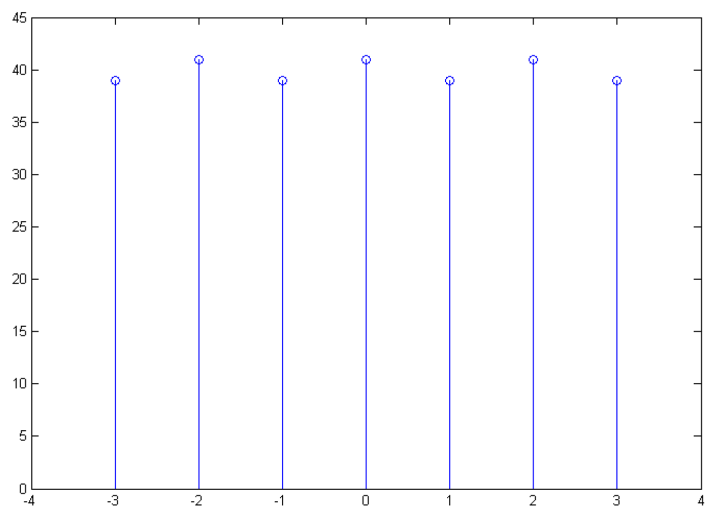
$$K\omega_0 < \omega_c$$
$$K < \frac{\omega_c}{\omega_0}$$

$$K = \lfloor \frac{\omega_c}{\omega_0} \rfloor = \lfloor \frac{4\pi/3}{\pi/2} \rfloor = \lfloor \frac{8}{3} \rfloor = 2$$

(e) Only $x(t)$.
$x(t)$ will exhibit Gibbs ringing because it has jump discontinuities.
$z(t)$, on the other hand, is the sum of a finite number of sinusoids.

5. Problem 5

(a) Let's call the first input $x_1[n]$, the second input $x_2[n]$, and the third input $x_3[n]$. To find the output of the system when the input is $x_3[n]$, let's represent it as a linear combination of $x_2[n]$ and $x_1[n]$. One possible such combination is $x_3[n] = 4x_1[n] - \frac{1}{5}x_2[n]$. By the properties of linearity, we can then represent our output as the same linear combination of $y_1[n]$ and $y_2[n]$. That is, our output is just $y_3[n] = 4y_1[n] - \frac{1}{5}y_2[n]$. This gives an output of



Another way to solve this problem involves frequency response. You can observe that the inputs can also be written as $x_1[n] = e^{i0n}$ and $x_2[n] = -10e^{i\pi n}$, and the outputs can be written as $y_1[n] = 10e^{i0n}$ and $y_2[n] = -5e^{i\pi n}$. Since the system is LTI, this implies that $H(0) =$

10 and $H(\pi) = 0.5$. Since the third output can be expressed as $x_3[n] = 4 + 2e^{i\pi n}$, the output must be $y_3[n] = H(0)\cdot 4 + H(\pi)\cdot 2e^{i\pi n} = 40 + e^{i\pi n}$. This is the same solution plotted above.

(b) We are given that $y[n] = a_1 x[n-1] + a_2 x[n-2]$. Let's use our two input-output pairs to figure out a_1 and a_2. From the first input-output pair, we see that $a_1 \cdot 1 + a_2 \cdot 1 = 10$. From the second pair, we look at $y[0]$ and see that $10a_1 - 10a_2 = -5$. This defines a matrix:

$$\begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 10 & -10 \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

We can solve this either using substitution or by inverting the matrix. Either way, the answer is that $a_1 = 4.75$ and $a_2 = 5.25$.

6. Problem 6: Ambulence Revisited

(a) We start by knowing that the time delay from the ambulence to the listner, $\tau(t)$ is a function of t. We also have the formula $distance = rate * time$, and $time = distance/rate$. Let v_s be the speed of sound, and $D(t)$ be the distance from the ambulence to the listener as a function of time.

$$y(t + \tau(t)) = x(t)$$
$$\tau(t) = \frac{D(t)}{v_s} = \frac{v_a t + d}{v_s}$$

We will now plug in $\tau(t)$ and use u-substitution to solve for the equation in the proper form.

$$x(t) = y(t + \frac{d + v_a t}{v_s}) = y(u)$$
$$u = t(1 + \frac{v_a}{v_s}) + \frac{d}{v_s}$$
$$t = \frac{u}{1 + \frac{v_a}{v_s}} - \frac{d}{v_s + v_a}$$
$$y(u) = x(t) = x(\frac{u}{1 + \frac{v_a}{v_s}} - \frac{d}{v_s + v_a})$$

Finally, we go back to the original form of the problem and compare.

$$y(t) = x(\frac{t}{a} - b)$$
$$a = 1 + \frac{v_a}{v_s} = 1.1$$
$$b = \frac{d}{v_s + v_a} = 1$$

(b) The system is not time invariant (it is time variant). Generally, time scaling is TV, but I'll prove it anyway.

$$\begin{aligned}
&y(t-\tau) = x(\frac{t-\tau}{a} - b) \\
&\hat{x}(t) = x(t-\tau) \\
&\hat{y}(t) = \hat{x}(\frac{t}{a} - b) = x(\frac{t}{a} - b - \tau) \\
&y(t-\tau) \neq \hat{y}(t)
\end{aligned}$$

A common explaination was that a change in frequency implies TV. However, that is not always true. Here's a counter example: $y(t) = (x(t))^2$.

(c) We have the time scaling factor, so we can simply apply this factor to the frequency of $x(t)$.

$$\begin{aligned}
&x(t) = cos(2\pi * 10^6 t) \\
&\omega = 2\pi * 10^6 \\
&f = \frac{\omega}{2\pi} = 10^6
\end{aligned}$$

If time is scaled by $\frac{1}{a}$, period is multiplied by a, and frequency is multiplied by $\frac{1}{a}$.

$$f_{new} = \frac{f_{old}}{a} = \frac{10}{11} * 10^6$$

This is called a doppler shift.

7. Problem 7

The period of $x(2t)$ is $T' = T/2$. Therefore $w_0' = 2w_0$. Solve for A_0':

$$A_0' = \frac{1}{T'} \int_0^{T'} x(2t)dt$$

Let $p = 2t$, $dp = 2dt$. Plug in:

$$A_0' = \frac{2}{T} \int_0^{T/2} x(2t)dt = \frac{2}{T} \int_0^{T} x(p)\frac{dp}{2} = \frac{1}{T} \int_0^{T} x(p)dp = A_0$$

Use the same process for A_k' and B_k'

$$A_k' = \frac{2}{T'} \int_0^{T'} x(2t)\cos(kw_0't)dt = \frac{4}{T} \int_0^{T/2} x(2t)\cos(k2w_0t)dt$$

Substitute $p = 2t$:

$$\frac{4}{T} \int_0^{T} x(p)\cos(k2w_0\frac{p}{2})\frac{dp}{2} = \frac{2}{T} \int_0^{T} x(p)\cos(kw_0p)dp = A_k$$

The same process will show that $B_k' = B_k$

8. Problem 8

   (a) $A_0 \neq 0,\ A_k = 0,\ B_k = 0$
   Any constant function only has an A_0 component. No sinusoidal components implies both A_k and B_k are zero.

   (b) $A_0 = 0,\ A_k = 0,\ B_k \neq 0$
   This square wave is odd and centered around zero. Any odd function only contains terms of B_k not equal to zero.

   (c) $A_0 = 0,\ A_k \neq 0,\ B_k \neq 0$
   This sinusoid has no DC offset, so A_0 will be 0. But since it is neither odd nor even, it will have terms A_k and B_k not equal to zero.

   (d) $A_0 \neq 0,\ A_k \neq 0,\ B_k = 0$
   The signal is strictly positive making A_0 not equal to zero. There is symmetry over the y-axis, making it even, and thus removing the odd B_k terms.

   (e) $A_0 = 0,\ A_k \neq 0,\ B_k \neq 0$
   This function has an average value of zero. Although it has some repetitive structure, the signal is neither odd nor even, thus making the A_k and B_k terms non-zero.

9. Problem 9
   The input-output relation of continuous-time LTI system F is given as

$$y(t) = 0.5y(t-1) + 0.3x(t)$$

   (a) Determine $F(\omega)$ and then plot $|F(\omega)|$.
   By the definition of Frequency Response, if the input is:

$$x(t) = e^{i\omega t}$$

   then the corresponding output is:

$$y(t) = F(\omega)e^{i\omega t}$$

   Also,

$$y(t-1) = F(\omega)e^{i\omega(t-1)} = F(\omega)e^{i\omega(-1)}e^{i\omega t} = e^{-i\omega}F(\omega)e^{i\omega t}$$

   Substituting these into the LCCDE, we get:

$$F(\omega)e^{i\omega t} = 0.5e^{-i\omega}F(\omega)e^{i\omega t} + 0.3e^{i\omega t}$$

   By dividing both sides by $e^{i\omega t}$ and rearranging the terms, we arrive at the following expression for $F(\omega)$:

$$F(\omega) = \frac{0.3}{1 - 0.5e^{-i\omega}}$$

In order to plot $|F(\omega)|$, take the maginitude of $F(\omega)$:

$$|F(\omega)| = |\frac{0.3}{1-0.5e^{-i\omega}}| = \frac{|0.3|}{|1-0.5e^{-i\omega}|} = \frac{0.3}{|1-0.5e^{-i\omega}|}$$

Using Euler's formula,

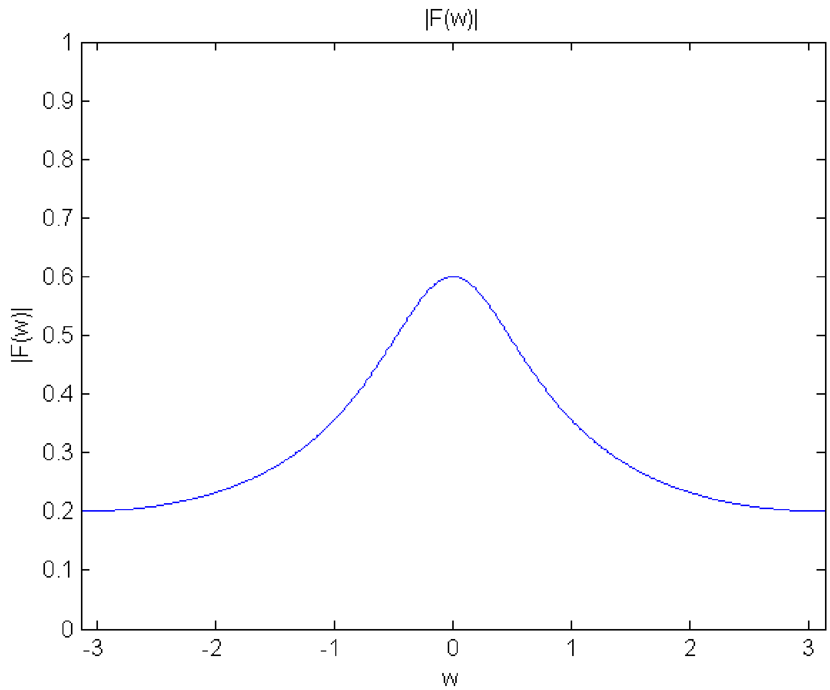
$$|F(\omega)| = \frac{0.3}{|1-0.5(cos(-\omega)+i \cdot sin(-\omega))|} = \frac{0.3}{|1-0.5cos(\omega)+0.5i \cdot sin(\omega)|}$$

Using Pythagorean theorem,

$$|F(\omega)| = \frac{0.3}{\sqrt{(1-0.5cos(\omega))^2+(0.5sin(\omega))^2}} = \frac{0.3}{\sqrt{1-cos(\omega)+0.25cos^2(\omega)+0.25sin^2(\omega))}}$$

which yields the following expression:

$$|F(\omega)| = \frac{0.3}{\sqrt{1.25-cos(\omega)}}$$



(b) If the input is $x(t) = 5\sqrt{2} + \frac{5}{3}\cos(\pi t - \frac{7\pi}{12})$, determine the corresponding output.
The expression, $y(t) = F(\omega)x(t)$ is only valid when $x(t)$ is a linear combination of complex exponentials. Expressing $x(t)$ as a linear combination of complex exponentials, we get:

$$x(t) = 5\sqrt{2} + \frac{5}{3}\cos(\pi t - \frac{7\pi}{12}) = 5\sqrt{2}e^{i0t} + \frac{5}{3}\frac{e^{i(\pi t - \frac{7\pi}{12})}}{2} + \frac{5}{3}\frac{e^{i-(\pi t - \frac{7\pi}{12})}}{2}$$

The corresponding output is:

$$y(t) = F(w)x(t) = H(w)(5\sqrt{2}e^{i0t} + \frac{5}{3}\cdot\frac{1}{2}e^{i(\pi t - \frac{7\pi}{12})} + \frac{5}{3}\cdot\frac{1}{2}e^{i-(\pi t - \frac{7\pi}{12})})$$

$$y(t) = F(w)x(t) = F(0)5\sqrt{2}e^{i0t} + F(\pi)\frac{5}{3}\cdot\frac{1}{2}e^{i(\pi t - \frac{7\pi}{12})} + F(-\pi)\frac{5}{3}\cdot\frac{1}{2}e^{i-(\pi t - \frac{7\pi}{12})}$$

Where

$$F(0) = \frac{0.3}{1 - 0.5e^{-i0}} = \frac{0.3}{1 - 0.5} = 0.6$$

$$F(\pi) = \frac{0.3}{1 - 0.5e^{-i\pi}} = \frac{0.3}{1 - 0.5(-1)} = 0.2$$

$$F(-\pi) = \frac{0.3}{1 - 0.5e^{i\pi}} = \frac{0.3}{1 - 0.5(-1)} = 0.2$$

Therefore,

$$y(t) = 0.6 \cdot 5\sqrt{2}e^{i0t} + 0.2 \cdot \frac{5}{3} \cdot \frac{1}{2}e^{i(\pi t - \frac{7\pi}{12})} + 0.2 \cdot \frac{5}{3} \cdot \frac{1}{2}e^{i-(\pi t - \frac{7\pi}{12})}$$

$$y(t) = 3\sqrt{2} + \frac{1}{3}\cos(\pi t - \frac{7\pi}{12})$$