1 Problem 1

1.1 A

Approach 1: We can calculate the electric field above both sheets using the principle of superposition. We will start by calculating the fields from only a charged sheet and a charged slab and then add them to get the total field. For only a charged sheet:

$$\oint \vec{E} \cdot d\vec{A} = E_{top}A - E_{bottom}A = \sigma A/\epsilon_0$$

+1 pt

and by symmetry we can say E_{top} and E_{bottom} are equal and opposite, yielding the contribution from the sheet:

$$\vec{E}_\sigma = -\frac{\sigma}{2\epsilon_0}\hat{z}$$

+1 pt

Likewise, we can make a similar argument for the electric field from a charged slab:

$$\vec{E}_\rho = +\frac{\rho D}{2\epsilon_0}\hat{z}$$

+1 pt

These can be added to get the total $\vec{E}$ field:

$$\vec{E}_{total} = \frac{\rho D}{2\epsilon_0}\hat{z} - \frac{\sigma}{2\epsilon_0}\hat{z} = \frac{\rho D - \sigma}{2\epsilon_0}\hat{z}$$

+2 pt

Approach 2: A gaussian pillbox including both the surface charge and the volume charge can be drawn around both surfaces. If we assert that the electric field above and below the charge distributions:

$$\oint \vec{E} \cdot d\vec{A} = E_{top}A - E_{bottom}A = ((\rho D - \sigma)A)/\epsilon_0$$

+ 3 pt

$\vec{E}_{top}$ and $\vec{E}_{bottom}$ are equal and opposite, so this yields the same result as superposition:

$$\vec{E}_{top} = \frac{\rho D - \sigma}{2\epsilon_0}\hat{z}$$

+ 2 pt

1.2 B

To find the electric field we use the principle of superposition. Above the point there is a total charge density:

$$\sigma_{above} = \rho d - \sigma$$

+1 pt

$$\sigma_{below} = \rho(D - d)$$

+1 pt

The total electric field is given by the principle of superposition;

$$\vec{E}_{total} = \vec{E}_{top} + \vec{E}_{bottom} = \frac{\sigma_{below}}{2\epsilon_0}\hat{z} - \frac{\sigma_{above}}{2\epsilon_0}\hat{z}$$

$$\vec{E}_{total} = \frac{\rho(D-d)}{2\epsilon_0}\hat{z} - \frac{\rho d - \sigma}{2\epsilon_0}\hat{z}$$

$$\vec{E}_{total} = \frac{\rho(D-2d)+\sigma}{2\epsilon_0}\hat{z}$$

+3 pt

1.3 C

We can find the electric field at any distance below both charges in the exact same way as we used to find the electric field above both charges, however the final result is pointed in the opposite direction. Using the same argument as A:

$$\vec{E}_{total} = \frac{\sigma - \rho D}{2\epsilon_0}\hat{z}$$

+5 pt

1.4 D

The electric fields above the positive sheet of charge is changed by:

$$\Delta E = \frac{\sigma}{2\epsilon_0}$$

+1 pt

and below the positive sheet of charge is changed by:

$$\Delta E = -\frac{\sigma}{2\epsilon_0}$$

+1 pt

We can now use the principle of superposition to modify the answer to parts A, B, C to find the new electric fields.

$$E_{top} = \frac{\rho D}{2\epsilon_0}\hat{z}$$

+1 pt

$$\vec{E}_{middle} = \frac{\rho(D-2d)+2\sigma}{2\epsilon_0}\hat{z}$$

+1 pt

$$\vec{E}_{below} = \frac{-\rho(D)}{2\epsilon_0}\hat{z}$$

+1 pt

Note: There are several valid ways to set up Gaussian surfaces to approach this problem. In the case of incorrect answers, partial credit was awarded generously based on the degree of understanding of Gauss's law demonstrated. Unsuccessful solutions attempting to use Coulomb's law were given no credit.

2.

(a)

Since there is no charge flows between two spheres, the electric potentials on two spheres are equal.

The electric potential on sphere A and B: $V_A = V_B$

(get 4 points for $V_A = V_B$ and reason)

Because D >> R_A and R_B

You can neglect the electric potential due to sphere B when calculate VA, and vice versa. (take 2 points without argument)

$$V_A = \frac{1}{4\pi\epsilon_o}\frac{Q_A}{R_A}$$

$$V_B = \frac{1}{4\pi\epsilon_o}\frac{Q_B}{R_B}$$

(get 4 points with correct V_A and V_B)

$$V_A = V_B \Rightarrow \frac{Q_A}{R_A} = \frac{Q_B}{R_B} \Rightarrow Q_A: Q_B = 3:1$$

$$Q_A + Q_B = 2 \times 10^{-8}\mathrm{C}$$

$$\Rightarrow Q_A = 1.5 \times 10^{-8}\mathrm{C}\ ,\ \ Q_B = 5 \times 10^{-9}\mathrm{C}$$

(two points for final answer)

(b)

Sphere A is connected to ground, thus the electric potential on sphere A is 0

The electric potential of sphere A is due to its own charge and the electric field due to sphere B. Since D >> R_A and R_B, you can take both spheres as point charges.

$$V_A = \frac{1}{4\pi\epsilon_o}\frac{Q_A'}{R_A} + \frac{1}{4\pi\epsilon_o}\frac{Q_B}{D} = 0$$

$$\Rightarrow Q_A' = -\frac{R_A}{D}Q_B = -\frac{1}{10}Q_B = -\frac{1}{30}Q_A$$

(get 4 points with V_A eq.)

(take 2 points without reasoning)

(get 1 point with correct Q_A')

Since sphere A now carries negative charge, there is electric force, F, attracting two spheres.

$$F_0 = \frac{1}{4\pi\epsilon_o}\frac{Q_A Q_B}{D} = 1 \times 10^{-5}\mathrm{N}$$

$$F = \frac{1}{4\pi\epsilon_o}\frac{Q_A' Q_B}{D} = \frac{1}{4\pi\epsilon_o}\frac{1}{30}\frac{Q_A Q_B}{D} = 3.33 \times 10^{-7}\mathrm{N}$$

(get three points with correct F eq.)

(get 2 points with correct answer, one for magnitude, one for direction)

(a) Note that since the area of a circular disc is πr^2, and the charge distribution is radially symmetric, I can write $dA = 2\pi r dr$. Since the charge distribution is uniform, Q is just a constant. I integrate from R_1 to R_2 since that is where the charge is. The total electric charge is:

$$Q = \int \sigma dA = \int_{R_1}^{R_2} \sigma 2\pi r dr = \sigma\pi(R_2^2 - R_1^2)$$

(b) The integral to evaluate is the following:

$$V = \int \frac{dq}{4\pi\epsilon_0 s}$$

Note that here, s is the distance from the element dq on the ring to the point where we want to find the electric field (where the point charge is). By geometry we get that:

$$s^2 = r^2 + z^2$$

Where r is the radial position of dq on the ring.

As before, we can rewrite $dq = \sigma dA$, and $dA = 2\pi r dr$,

$$\begin{aligned} V &= \int \frac{\sigma dA}{4\pi\epsilon_0\sqrt{r^2+z^2}} \\ &= \int_{R_1}^{R_2} \frac{\sigma 2\pi r dr}{4\pi\epsilon_0\sqrt{r^2+z^2}} \\ &= \int_{R_1^2+z^2}^{R_2^2+z^2} \frac{\sigma du}{4\epsilon_0\sqrt{u}} \\ &= \frac{\sigma}{2\epsilon_0}(\sqrt{R_2^2+z^2} - \sqrt{R_1^2+z^2}) \end{aligned}$$

In line 3 I used the substitution $u = r^2 + z^2$.

(c) Look at the limit that $z \ll R_1, R_2$. Then I can write:

$$V = \frac{\sigma}{2\epsilon_0}\left(R_2\sqrt{1+\left(\frac{z}{R_2}\right)^2} - R_1\sqrt{1+\left(\frac{z}{R_1}\right)^2}\right)$$

Using the expansion $(1+x)^\alpha \approx 1 + \alpha x$ from the equation sheet:

$$V \approx \frac{\sigma}{2\epsilon_0}\left(R_2 + \frac{z^2}{2R_2} - R_1 - \frac{z^2}{2R_1}\right) = \frac{\sigma}{2\epsilon_0}\left(R_2 - R_1 + \left(\frac{1}{2R_2} - \frac{1}{2R_1}\right)z^2\right)$$

So the potential is quadratic in position. (Note that the constant term does not matter as the place where potential is 0 could be moved to a place so that that constant is zero).

(d) The energy of the negative charge when it is displaced from the center of the ring is:

$$\Delta U = -q\Delta V = -q(V(z) - V(0)) \approx \frac{q\sigma(R_2 - R_1)}{4\epsilon_0 R_2 R_1} z^2$$

The simplest way to solve this is to realize that this is just the energy due to a spring with $k = \frac{q\sigma(R_2-R_1)}{2\epsilon_0 R_2 R_1}$. Then, using the hint you can write:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{q\sigma(R_2 - R_1)}{2m\epsilon_0 R_2 R_1}}$$

If you did not remember this, there is still hope. You know that energy is conserved, so while oscillating it must be true that:

$$\frac{q\sigma(R_2 - R_1)}{4\epsilon_0 R_2 R_1} z(t)^2 + \frac{m}{2} z'(t)^2 = \text{const.}$$

In an oscillation, the particle is travelling in a periodic motion, so you could write that $z(t) = 0.01 R_1 \cos(\omega t)$. Plugging this in, you see that you have something that looks like $C\cos^2(\omega t) + D\sin^2(\omega t) = \text{const}$. This can only be true if $C = D$, so solving that for ω with this gives the same answer as before.

Yet another solution is to take the time derivative of the energy conservation equation above. This gives:

$$\frac{q\sigma(R_2 - R_1)}{2\epsilon_0 R_2 R_1} z(t) z'(t) + m z''(t) z'(t) = 0$$

In general, $z'(t)$ is nonzero so I can divide by it to get:

$$m z''(t) = -\frac{q\sigma(R_2 - R_1)}{2\epsilon_0 R_2 R_1} z(t)$$

This is a differential equation on the equation sheet for which the solution is given, and you could pick out the frequency from that solution.

Midterm 2, Question 3 (Dan and Alison)

a) (4 points)
   - 2 points for understanding that $Q=\sigma A$, but having the wrong expression for A (need to actually plug something in for A, can't just say A)
   - -1 for algebra mistake

b) (6 points)
   - +1 point for outlining correct methodology to find V (either direct integral form OR finding E then integrating)
     - NOTE: just writing down the formula isn't sufficient to get any points. You must somehow reference geometry of the situation, or indicate you know how you will evaluate the integrals
   - +1 for rewriting $dq=\sigma dA$
   - +1 for $s^2=r^2+x^2$
   - +1 for rewriting $dA=2\pi r dr$
   - +2 for correctly following through with outlined methodology, i.e. taking the integrals correctly (provided you show a correct way to calculate E, if you calculate V from E)
   - -1 for algebra mistake, writing down slightly wrong equation, other minor errors
   - NOTE: maximum 1 point for entire part (b) if either Gauss' law is used OR you consider the disk as a point charge (represents fundamental misunderstanding of geometry of problem)
   - NOTE: max 3 points for claiming electric potential has components and making an argument like that

c) (5 points)
   - +1 for realizing you need to take the limit as $x \to 0$
   - +3 for using correct Taylor approximation
   - +1 for recognizing quadratic dependence (assuming your taylor expansion was correct)
   - -1 for algebra mistake
   - NOTE: max 2 if right taylor expansion, but used the wrong potential

d) (5 points)
   - +4 points regardless of method (energy/force) relating known quantities to a equation of k and ω (correctly)
   - +1 point for correct final answer
   - If equation is incorrect or not present:
     - +2 maximum for outlining a correct method to find harmonic dependence (could be only +1 if not enough detail is present), this could be if you don't get F proportional to x etc.

1 Problem 4

1.1 Part a

Initially the second capacitor acts like a wire and we see a voltage $V = Q/C_1$ being discharged through the resistor. So $I = \frac{V}{R} = \frac{Q}{C_1 R}$

2 Points for the voltage and 2 points for the current

1.2 Part b

Charge must be conserved and the two will have equal and opposite voltages.

$$\begin{aligned}
\frac{Q}{C_1} &= \frac{Q_0 - Q}{C_2} \\
\frac{1}{C_1} + \frac{1}{C_2} &= \frac{1}{C_{eq}} \\
\frac{Q}{C_{eq}} &= \frac{Q_0}{C_2} \\
Q &= \frac{C_{eq}}{C_2} Q_0 \\
&= \frac{C_1 Q_0}{C_1 + C_2}
\end{aligned}$$

1 point for charge conservation. 1 point for the voltages being equal. 2 points to solve the system.

1.3 Part c

Write down the loop rule

$$\begin{aligned}
\frac{Q}{C_1} + \frac{dQ}{dt} R - \frac{Q_0 - Q}{C_2} &= 0 \\
\frac{dQ}{dt} &= -\frac{1}{R} Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) + \frac{Q_0}{RC_2} \\
a &= \frac{1}{RC_eq} \\
b &= -\frac{Q_0}{RC_2} \\
\frac{dQ}{aQ + b} &= -dt
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{a} ln \frac{aQ+b}{b} &= -t \\
Q &= \frac{b}{a} e^{-at} - \frac{b}{a} \\
&= \frac{Q_0 C_1}{C_1 + C_2} (e^{-(C_1+C_2)t/RC_1C_2} - 1)
\end{aligned}
$$

It is 0 initially and the answer in part b at ∞ as desired.

3 points for writing down the loop rule correctly. 2 points to solve the differential equation.

1.4 Part d

The current is

$$
\begin{aligned}
I &= \frac{dQ}{dt} \\
&= \frac{Q_0}{RC_2} e^{-(C_1+C_2)t/RC_1C_2}
\end{aligned}
$$

+1 point for saying it is the derivative. +3 points for differentiating correctly. 2 if the result showed inconsistensies.

1.5 Part e

If a dielectric is inserted into the second capacitor, it's capacitance changes by a factor $\frac{\epsilon}{\epsilon_0}$

Inserting that gives the new Q_2

$$
= \frac{Q_0 C_1}{C_1 + \kappa C_2} (e^{-(C_1+\kappa C_2)t/RC_1\kappa C_2} - 1)
$$

2 points for using ϵ instead.

Prob 5

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(a)

Consider a dipole consisted of two point charges $-q$ and q separated by distance d. Set up a coordinate system so that these two charges lie on the z axis. $-q$ lies at $\mathbf{r}_- = (0,\ 0,\ -d/2)$, and q at $\mathbf{r}_+ = (0,\ 0,\ d/2)$. At a point $\mathbf{r} = (0,\ 0,\ z)$ on the axis. Then the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{-q\hat{\mathbf{z}}}{\mid \mathbf{r}-\mathbf{r}_- \mid^2} + \frac{1}{4\pi\epsilon_0}\frac{q\hat{\mathbf{z}}}{\mid \mathbf{r}-\mathbf{r}_+ \mid^2} \\
&= \frac{q}{4\pi\epsilon_0}\left[-\left(z+\frac{d}{2}\right)^{-2} + \left(z-\frac{d}{2}\right)^{-2}\right]\hat{\mathbf{z}} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{z^2}\left[-\left(1+\frac{d}{2z}\right)^{-2} + \left(1-\frac{d}{2z}\right)^{-2}\right]\hat{\mathbf{z}} \qquad (1)
\end{aligned}$$

When the observation point is far from the dipole, $z \gg d$, the above expression can be Taylor expanded (recall that $(1+x)^a \approx 1 + ax$ for $x \ll 1$)

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{1}{z^2}\left[-\left(1+\frac{d}{2z}\right)^{-2} + \left(1-\frac{d}{2z}\right)^{-2}\right]\hat{\mathbf{z}} \\
&\approx \frac{q}{4\pi\epsilon_0}\frac{1}{z^2}\left[-\left(1+\frac{d}{2z}\times(-2)\right) + 1 - \frac{d}{2z}\times(-2)\right]\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\,(qd)\,\hat{\mathbf{z}}}{z^3} \qquad (2)
\end{aligned}$$

Identifying the dipole moment in the above expression: $\mathbf{p} \equiv q\mathbf{d} = qd\hat{\mathbf{z}}$:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\mathbf{p}}{z^3} \qquad (3)$$

Comment: $\mathbf{p} \equiv q\mathbf{d} = qd\hat{\mathbf{z}}$, in the above setting, follows the convention in physics, that a dipole's dipole moment points from the negative charge to the positive charge. Chemists sometimes use a different convention, with dipole moment points in the opposite direction $\mathbf{p} \equiv q\mathbf{d} = -qd\hat{\mathbf{z}}$. Hence if you follow the Chemistry convention, your energy formula will take the form $U = \mathbf{p}\cdot\mathbf{E}$ rather than $U = -\mathbf{p}\cdot\mathbf{E}$. We are rather loose this time about the convention, though you are expected to state it explicitly, if you ever realize. However, only partial credit will be given for having wrong signs in part (b) and (c).

(b)

If the two dipoles are aligned, $\mathbf{p}_1 \cdot \mathbf{p}_2 = p_1 p_2$, hence,

$$\begin{aligned} U &= -\mathbf{p}_2 \cdot \mathbf{E}_1 \\ &= -\mathbf{p}_2 \cdot \left(\frac{1}{4\pi\epsilon_0} \frac{2\mathbf{p}_1}{r^2} \right) \\ &= -\frac{1}{4\pi\epsilon_0} \frac{2p_1 p_2}{r^3} \end{aligned} \tag{4}$$

where $\mathbf{E}_1$ is the electric field produced by $\mathbf{p}_1$.

> Comment: No points will be given for simply put down $U = -\mathbf{p} \cdot \mathbf{E}$, as the formula itself makes no sense unless you put it in a context. At most 1 points would be given if you simply claim that $\mathbf{E}$ from (a) should be plugged in $U = -\mathbf{p} \cdot \mathbf{E}$. You have to show how you adapt the formula in (a) to reflects the change of configuration.

(c)

If the two dipoles are anti-aligned, $\mathbf{p}_1 \cdot \mathbf{p}_2 = -p_1 p_2$, hence,

$$\begin{aligned} U &= -\mathbf{p}_2 \cdot \mathbf{E}_1 \\ &= -\frac{1}{4\pi\epsilon_0} \frac{2\mathbf{p}_1}{r^2} \cdot \mathbf{p}_2 \\ &= \frac{1}{4\pi\epsilon_0} \frac{2p_1 p_2}{r^3} \end{aligned} \tag{5}$$

(d)

Inside an electric field, a dipole feels no net electric force. But it does experience a torque $\boldsymbol{\tau}$

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \tag{6}$$

which has magnitude

$$\tau = pE \sin\theta \tag{7}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{E}$.

In the figure below the part (d), the dipoles have their dipole moments either parallel or antiparallel to the electric field, giving $\sin\theta = 0$. Hence the torque vanish for both dipoles. There is no net force, either. So the dipoles will remain their positions and configurations.

> Comment: $\mathbf{p}_1$ is in a stable equilibrium while $\mathbf{p}_2$ an unstable equilibrium, susceptible to tiny purturbation. Hence unless you explicitly mention that $\mathbf{p}_2$ is unstable, we will not credit answers saying that $\mathbf{p}_2$ will flip its direction.
> Again if you adopted the chemists convention of dipole of moment, $\boldsymbol{\tau} = -\mathbf{p} \times \mathbf{E}$. And it is $\mathbf{p}_2$ in stable equilibrium.

For a dipole in an electric field, the torque is maximum when $\mathbf{p} \perp \mathbf{E}$, and $\tau_{max} = pE$. This result comes straightfowardly from Eq. (7).

Problem 5 Rubric

a.

Find electric field of the point charges at appropriate locations and add them	3
Taylor expand in the appropriate limit	3
Introduce dipole moment	1
Final answer	1

b.

-2 for sign error (unless consistent with dipole moment definition)

c.

All points if consistent with part b (non-trivially)

d.

What happens? Nothing. What is the torque? Zero.	3
Maximum torque configuration?	2
Maximum torque?	1