1 Problem 1

a).

$$h[n] = \begin{cases} -0.25 & \text{if } n = 0 \\ 0.5 & \text{if } n = 1 \\ -0.25 & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The system is <u>causal</u> since $h[n] = 0$ for all $n < 0$. The system is <u>stable</u> since

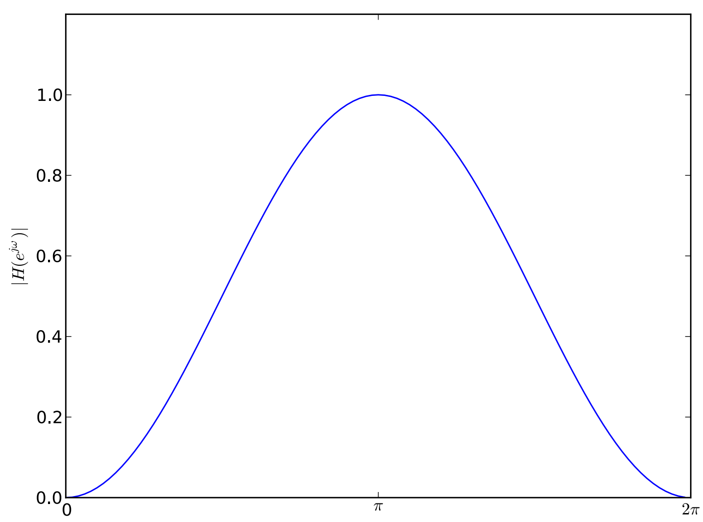
$$\sum_{n=-\infty}^{\infty} |h[n]| = 0.25 + 0.5 + 0.25 = 1 < \infty. \tag{2}$$

b).

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} && (3) \\ &= -0.25 + 0.5e^{-j\omega} - 0.25e^{-2j\omega} && (4) \\ &= e^{-j\omega}(-0.25e^{j\omega} + 0.5 - 0.25e^{-j\omega}) && (5) \\ &= e^{-j\omega}(0.5 - 0.5\cos(\omega)) && (6) \\ &= e^{-j\omega}(\sin^2(\omega/2)) \qquad \text{(alternate answer)} && (7) \end{aligned}$$

$$\begin{aligned} |H(e^{j\omega})| &= (0.5 - 0.5\cos(\omega)) && (8) \\ &= (\sin^2(\omega/2)) && (9) \end{aligned}$$

Highpass Filter



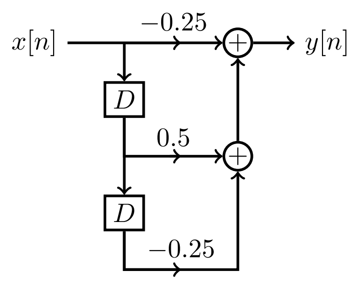
Comments:

- It was expected to see one of (6) or (7) or (8) or (9), *i.e.* some simplification of the impulse response or impulse response magnitude to sine/cosine form was necessary.
- The plot does not have cusps or corners at $0, 2\pi$, *etc.* It is a simple cosine shifted up (no deducted points)

c). The system is linear phase: $A(e^{j\omega}) \triangleq (0.5 - 0.5\cos(\omega))$ is real and nonnegative, and $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega}$
Comments:

- Two points for correct answer, three points for correct justification/explanation/derivation
- Just because a function has a cosine or sine does not mean it automatically takes on positive and negative values

d).



2 Problem 3

a). $X(e^{j0}) = 4 - 1 + 3 - 2 + 3 - 1 + 4 = 10$

b). $X(e^{j\pi}) = 4 + 1 + 3 + 2 + 3 + 1 + 4 = 18$

c). $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = 8\pi$.

d). $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=0}^{6} |x[n]|^2 = 112\pi$

e). Note that $\tilde{x}[n] \triangleq x[n+3]$ is even symmetric, thus the transform of $\tilde{x}[n]$ is real. By the time shifting property of Fourier Transforms, $e^{-3j\omega}A(e^{j\omega})$ is the Fourier Transform of $x[n]$ where $A(e^{j\omega})$ is the (real) Fourier Transform of $\tilde{x}[n]$. We then have that $\measuredangle(X(j\omega)) = -3\omega$ and thus $\frac{d}{d\omega}\measuredangle X(e^{j\omega}) = -3$.

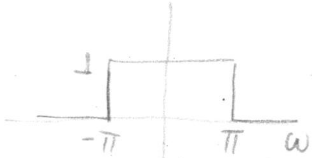
2. (20 points) Use the convolution property of Fourier transforms to prove the following identities for the function:

$$\mathrm{sinc}(t) \triangleq \frac{\sin \pi t}{\pi t}.$$

a) (10 points) $\mathrm{sinc}(t) * \mathrm{sinc}(t) = \mathrm{sinc}(t)$.

b) (10 points) $\mathrm{sinc}(t) * \sin(\omega_0 t) = \sin(\omega_0 t)$ if $\omega_0 < \pi$.

From Lec. 4 notes: $\mathrm{sinc}(t) \triangleq \frac{\sin \pi t}{\pi t} \xleftrightarrow{\mathcal{F}}$

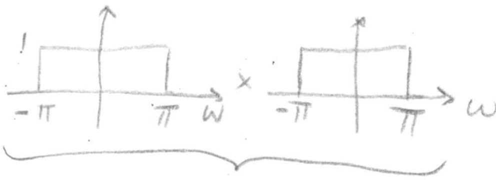


$= \begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| \geq \pi \end{cases}$

a) Use convolution property: $x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(j\omega)Y(j\omega)$

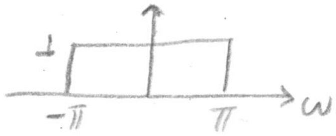
Then

$\mathrm{sinc}(t) * \mathrm{sinc}(t) \xrightarrow{\mathcal{F}}$

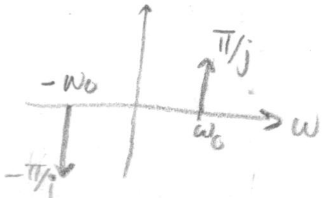


$\|$

$\mathrm{sinc}(t) \triangleq \frac{\sin \pi t}{\pi t} \xleftarrow{\mathcal{F}^{-1}}$

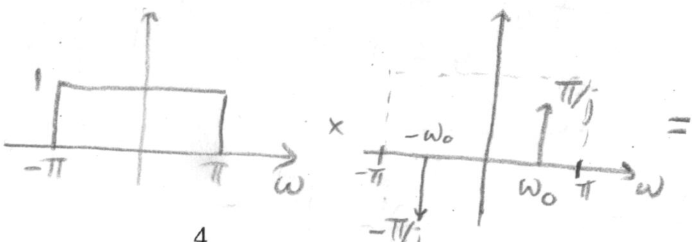


b) Note that $\sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \xrightarrow{\mathcal{F}}$



Convolution Property:

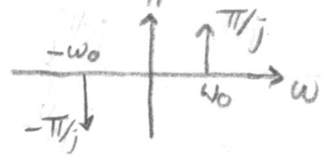
$\mathrm{sinc}(t) * \sin(\omega_0 t) \xrightarrow{\mathcal{F}}$



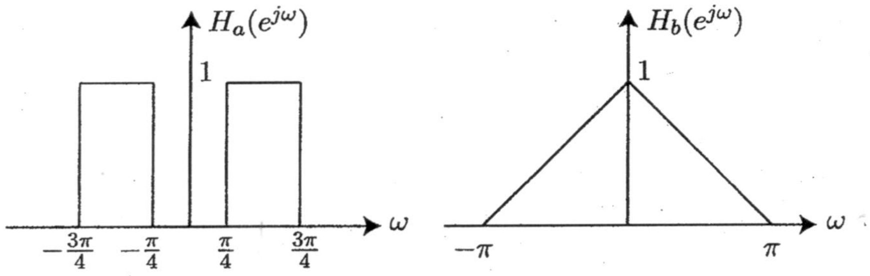
Since $\omega_0 < \pi$

$\|$

$\sin(\omega_0 t) \xleftarrow{\mathcal{F}^{-1}}$

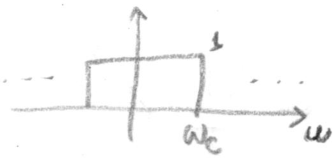


4. (20 points) The frequency response for two discrete-time filters are depicted below for $\omega \in [-\pi, \pi]$. Determine the impulse response of each filter.

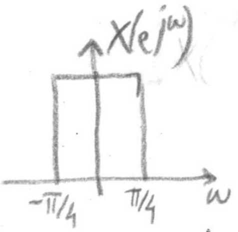


(a) (b)

From Lecture notes: $\dfrac{\sin(\omega_c n)}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow}$



a) First note



$\xrightarrow{\mathcal{F}^{-1}} x[n] = \dfrac{\sin(\frac{\pi}{4}n)}{\pi n}$

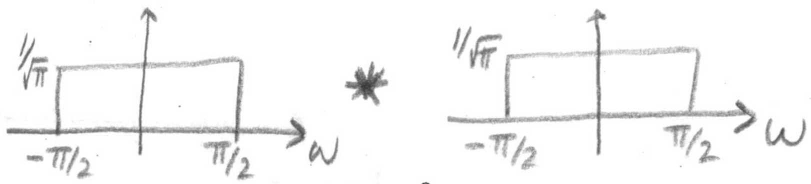
$$H_a(e^{j\omega}) = X(e^{j(\omega-\pi/2)}) + X(e^{j(\omega+\pi/2)})$$

$\downarrow \mathcal{F}^{-1}$

$$h_a[n] = e^{j\frac{\pi}{2}n} x[n] + e^{-j\frac{\pi}{2}n} x[n] = \frac{2\cos(\frac{\pi}{2}n)\sin(\frac{\pi}{4}n)}{\pi n}$$

b) Note that $H_b(e^{j\omega})$ is triangular and can be written as the convolution of two square waves:

$H_b(e^{j\omega}) =$



Then

$$h_b[n] = 2\pi\left(\frac{1}{\sqrt{\pi}}\frac{\sin(\pi/2\, n)}{\pi n}\,\frac{1}{\sqrt{\pi}}\frac{\sin(\pi/2\, n)}{\pi n}\right)$$

$$= 2\left(\frac{\sin(\pi/2\, n)}{\pi n}\right)^2 = \begin{cases} 1/2, & n=0 \\ 0, & n \text{ even} \\ 2/\pi^2 n^2, & n \text{ odd} \end{cases}$$

[Convolution in Freq. Domain: $\frac{1}{2\pi}\left(X(e^{j\omega}) * Y(e^{j\omega})\right) \updownarrow x[n]y[n]$]

5. a) (10 points) Find the two-dimensional CTFT for the signal:

$$x(t_1, t_2) = e^{-at_1} e^{-bt_2} u(t_1) u(t_2).$$

How should the real numbers a and b restricted for the CTFT to exist?

b) (10 points) Suppose the two-dimensional DTFT for $x[n_1, n_2]$ is given by $X(e^{j\omega_1}, e^{j\omega_2})$. Derive an expression for the one-dimensional DTFT of:

$$x_0[n_1] \triangleq \sum_{n_2=-\infty}^{\infty} x[n_1, n_2]$$

in terms of $X(e^{j\omega_1}, e^{j\omega_2})$.

a) $X(jw_1, jw_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-at_1} e^{-bt_2} e^{-jw_1 t_1} e^{-jw_2 t_2} u(t_1) u(t_2)\, dt_1\, dt_2$ (Analysis Eq.)

Seperability $x(t_1,t_2) = x_1(t_1) x_2(t_2)$

$= \underbrace{\int_0^{+\infty} e^{-at_1} e^{-jw_1 t_1} dt_1}\, \times \underbrace{\int_0^{+\infty} e^{-bt_2} e^{-jw_2 t_2} dt_2}$

$= \left[-\frac{1}{a+jw_1} e^{-(a+jw_1)t_1}\right]_0^{\infty} \times \left[-\frac{1}{b+jw_2} e^{-(b+jw_2)t_2}\right]_0^{\infty}$

* Only defined (limit $t\to\infty$) if $a>0$, so that magnitude $e^{-at_1} \to 0$ as $t_1 \to \infty$)
* Similarly for b.

Then

$X(jw_1, jw_2) = \frac{1}{a+jw_1} \cdot \frac{1}{b+jw_1}$ for $a>0$ and $b>0$

b) Note that (Analysis Eq. 2D-DTFT): $X(e^{jw_1}, e^{jw_2}) = \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} x[n_1,n_2] e^{-jw_1 n_1} e^{-jw_2 n_2}$

And from Analysis Eq. 1D-DTFT:

$X_0(e^{jw}) = \sum_{n_1=-\infty}^{+\infty} x_0[n_1] e^{-jwn_1}$

$\triangleq \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} x[n_1,n_2] e^{-jwn_1} \times e^{j0} = X(e^{jw}, e^{j0})$

Therefore $X_0(e^{jw}) = X(e^{jw_1}, e^{jw_2})\big|_{w_2=0}$ (Projection-Slice Theorem)