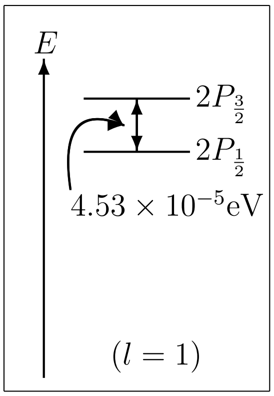
Physics 137B (Quantum Mechanics II) – Midterm Exam / October 15, 2013 / 9:40am - 11:00am

Please solve all three problems below, and please explain your answers.

Problem 1 [30pts]: Spin-orbit interaction



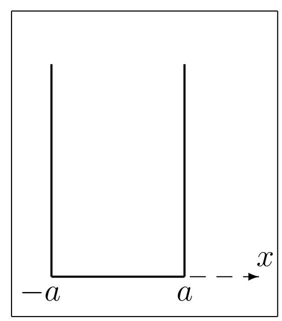
The spin-orbit interaction $H'_{\text{so}} = \left(\frac{e^2}{8\pi\epsilon_0}\right)\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}$ causes the degenerate $2P$ level of the hydrogen atom (the level with principal quantum number $n = 2$ and orbital quantum number $l = 1$) to split into two levels: $2P_{\frac{1}{2}}$ (with total angular momentum quantum number $j = \frac{1}{2}$) and $2P_{\frac{3}{2}}$ (with total angular momentum quantum number $j = \frac{3}{2}$). They differ in energy by

$$\Delta E = E(2P_{\frac{3}{2}}) - E(2P_{\frac{1}{2}}) = 4.53 \times 10^{-5}\text{eV}\,.$$

Hypothetically, if the electron had spin $s = 1$ (instead of $s = \frac{1}{2}$), assuming the same form of H'_{so}, into how many distinct levels would $2P$ split, and what would be the energy differences between adjacent levels? (You can express your answers in terms of ΔE.)

Problem 2 [40pts]: Fermions and perturbation theory.

Two identical and non-interacting fermions of mass m are in an infinite square well potential in the range $-a < x < a$. We ignore the spin in this problem (assuming that both particles are in the same spin state).



(a) What is the energy of the ground state of the system?

(b) Write down the wavefunction $\psi(x_1, x_2)$ of the ground state of the system.

(c) A perturbation $H' = \lambda x_1 x_2$ is added to the Hamiltonian. Calculate the first order $O(\lambda)$ correction to the ground state energy.

You can use one or more of the formulas

$$\int_{-a}^{a}\cos^2(\frac{\pi x}{2a})dx = \int_{-a}^{a}\sin^2(\frac{\pi x}{a})dx = a\,,$$

$$\int_{-a}^{a}x\cos(\frac{\pi x}{2a})\sin(\frac{\pi x}{a})dx = \frac{32a^2}{9\pi^2}\,,\qquad \int_{-a}^{a}x\sin^2(\frac{\pi x}{a})dx = \int_{-a}^{a}x\cos^2(\frac{\pi x}{2a})dx = 0\,,$$

$$\int_{-a}^{a}x^2\sin^2(\frac{\pi x}{a})dx = \frac{a^3}{3} - \frac{a^3}{2\pi^2}\,,\qquad \int_{-a}^{a}x^2\cos^2(\frac{\pi x}{2a})dx = \frac{a^3}{3} - \frac{2a^3}{\pi^2}\,,$$

(some formulas where added to hide the really needed one.)

Please turn the page . . . . $\Longrightarrow$

Problem 3 [30pts]: quick one-line answers.

Please answer the following questions and add to your answers one-line explanations.

(a) Two spin-$\frac{3}{2}$ particles are in a state $|\frac{3}{2}, \frac{3}{2}\rangle$ where both z-components of spin have eigenvalues $+\frac{3}{2}\hbar$. What would a measurement of the square of the total spin $(\vec{S}_1 + \vec{S}_2)^2$ give?

(b) Two identical fermions have an orbital wavefunction

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\pi b^3} e^{-(r_1+r_2)/b} .$$

Considering the following possibilities for their spin-part of the wavefunction:

$$(1) \quad |\uparrow\uparrow\rangle; \qquad (2) \quad |\uparrow\downarrow\rangle; \qquad (3) \quad |\downarrow\uparrow\rangle; \qquad (4) \quad |\downarrow\downarrow\rangle;$$

pick one of the following choices:

(A) Exactly one answer from (1)-(4) is correct. [Which ones is it?];
(B) there is more than one correct answer (1)-(4). [Which ones are correct?];
(C) none of the answers (1)-(4) is correct.

(c) Taking into account fine-structure, one finds that the lowest 6 energy levels of the hydrogen atom are

$$E_1 = -13.60587\text{eV}, \quad E_2 = -3.401480\text{eV}, \quad E_3 = -3.401434\text{eV},$$
$$E_4 = -1.511763\text{eV}, \quad E_5 = -1.511750\text{eV}, \quad E_6 = -1.511746\text{eV}$$

Write down the multiplicities (degeneracies) of each level. (Ignore the hyperfine splitting and the factor of 2 due to the spin of the proton.)

You may wish to recall the formula

$$E_{nj} = -\frac{1}{2n^2} mc^2\alpha^2 + \frac{1}{n^4}\left(\frac{3}{4} - \frac{n}{j+\frac{1}{2}}\right) mc^2\alpha^4 , \qquad \left(\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.035999173(35)}\right)$$

Solution to Problem 1

For given n and l, the radial part $R_{nl}(r)$ of the wavefunction is fixed. Denote the constant

$$C_{nl} \equiv \langle \left(\frac{e^2}{8\pi\epsilon_0} \right) \frac{1}{m^2c^2r^3} \rangle = \int \left(\frac{e^2}{8\pi\epsilon_0} \right) \frac{1}{m^2c^2r^3} R_{nl}(r)^2 4\pi r^2 dr$$

Then, the shift of the various $2P$ levels is given by

$$\Delta E_{nlj} = C_{nl} \langle \vec{S} \cdot \vec{L} \rangle = C_{nl} \langle \tfrac{1}{2}(J^2 - S^2 - L^2) \rangle = \tfrac{1}{2} C_{nl} \hbar^2 [j(j+1) - l(l+1) - s(s+1)]$$

For $s = \frac{1}{2}$ (using the fact that $|l-s| \leq j \leq l+s$) we have

$$j(j+1) - l(l+1) - s(s+1) = \begin{cases} (l+\frac{1}{2})(l+\frac{3}{2}) - l(l+1) - \frac{3}{4} = l = 1 & \text{for } j = l + \frac{1}{2} \\ (l-\frac{1}{2})(l+\frac{1}{2}) - l(l+1) - \frac{3}{4} = -1 - l = -2 & \text{for } j = l - \frac{1}{2} \end{cases}$$

So

$$\Delta E = E(2P_{\frac{3}{2}}) - E(2P_{\frac{1}{2}}) = \tfrac{3}{2} \hbar^2 C_{nl}$$

Again, recall, for $j = l + s$, we can have values of j that range from $|l-s| \leq j \leq l+s$. Therefore, for $s = 1$ we would have had (for $l = 1$ we get $|l-s| \leq j \leq l+s; \Rightarrow |1-1| \leq j \leq 1+1; \Rightarrow \boxed{0 \leq j \leq 2}$):

$$j(j+1) - l(l+1) - s(s+1) = \begin{cases} (l+1)(l+2) - l(l+1) - 2 = 2l = 2 & \text{for } j = l+1 \\ l(l+1) - l(l+1) - 2 = -2 & \text{for } j = l \\ l(l-1) - l(l+1) - 2 = -2 - 2l = -4 & \text{for } j = l-1 \end{cases}$$

So

$$E(2P_2) - E(2P_1) = 2\hbar^2 C_{nl} = \tfrac{4}{3} \Delta E \,, \qquad E(2P_1) - E(2P_0) = \hbar^2 C_{nl} = \tfrac{2}{3} \Delta E \,.$$

Solution to Problem 2

(a) The ground state wavefunction for a single particle is

$$\psi_0(x) = \frac{1}{\sqrt{a}} \cos(\frac{\pi x}{2a}) \,, \qquad \text{with energy } E_0 = \frac{\hbar^2 \pi^2}{8ma^2} \,,$$

and the first excited state is

$$\psi_1(x) = \frac{1}{\sqrt{a}} \sin(\frac{\pi x}{a}) \,, \qquad \text{with energy } E_1 = \frac{\hbar^2 \pi^2}{2ma^2} \,.$$

In the ground state of the two-fermion system, one fermion occupies ψ_0 and the other occupies ψ_1, so the total energy is

$$E = E_0 + E_1 = \frac{5\hbar^2 \pi^2}{8ma^2} \,.$$

(b)

$$\psi(x_1, x_2) = \tfrac{1}{\sqrt{2}} [\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)] = \frac{1}{\sqrt{2}a} [\cos(\frac{\pi x_1}{2a}) \sin(\frac{\pi x_2}{a}) - \cos(\frac{\pi x_2}{2a}) \sin(\frac{\pi x_1}{a})]$$

(c) We need first order perturbation theory:

$$E_0^{(1)} = \lambda\langle\psi|x_1x_2|\psi\rangle = \lambda\int_{-a}^{a}\int_{-a}^{a} x_1x_2|\psi(x_1,x_2)|^2dx_1dx_2 = \lambda(A+B+C).$$

$$\begin{aligned} A &= \frac{1}{2a^2}\int_{-a}^{a}\int_{-a}^{a} x_1x_2\cos^2(\frac{\pi x_1}{2a})\sin^2(\frac{\pi x_2}{a})dx_1dx_2 = 0 \\ B &= -\frac{1}{a^2}\int_{-a}^{a}\int_{-a}^{a} x_1x_2\sin(\frac{\pi x_1}{a})\cos(\frac{\pi x_1}{2a})\sin(\frac{\pi x_2}{a})\cos(\frac{\pi x_2}{2a})dx_1dx_2 = -\frac{1}{a^2}\left(\frac{32a^2}{9\pi^2}\right)^2 = -\frac{1024a^2}{81\pi^4}\,, \\ C &= \frac{1}{2a^2}\int_{-a}^{a}\int_{-a}^{a} x_1x_2\cos^2(\frac{\pi x_2}{2a})\sin^2(\frac{\pi x_1}{a})dx_1dx_2 = 0\,. \end{aligned}$$

So,

$$E_0^{(1)} = -\frac{1024a^2}{81\pi^4}\lambda.$$

Solution to Problem 3

(a)

$$j = \tfrac{3}{2}+\tfrac{3}{2} = 3 \Longrightarrow (S_1+S_2)^2 \to \hbar^2 j(j+1) = 12\hbar^2.$$

(b) (C) Up to phase, the spin part has to be a singlet $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

(c) The multiplicity of the a level with total angular momentum j is $(2j+1)$, and we have to add the $(2j+1)$'s for all possible l's, so (also, using the fact that $|l-s| \le j \le l+s$) we get:

$$E_1 = E(1S_{\frac{1}{2}}) \to 2, \quad E_2 = E(2S_{\frac{1}{2}}) = E(2P_{\frac{1}{2}}) \to 2+2 = 4, \quad E_3 = E(2P_{\frac{3}{2}}) \to 4,$$

$$E_4 = E(3S_{\frac{1}{2}}) = E(3P_{\frac{1}{2}}) \to 2+2 = 4, \quad E_5 = E(3P_{\frac{3}{2}}) = E(3D_{\frac{3}{2}}) \to 4+4 = 8, \quad E_6 = E(3D_{\frac{5}{2}}) = 6.$$