First Midterm Examination
Monday September 30 2013
Closed Books and Closed Notes
Answer Both Questions

Question 1
A Particle on a Rotating Plate
20 Points

As shown in Figure 1, a particle of mass m is attached to a fixed point O by a linearly elastic spring. The spring has a stiffness K and an unstretched length ℓ_0. The particle is free to move on a smooth groove on a disk which rotates about the vertical axis with a speed $\Omega(t)$. A vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle.

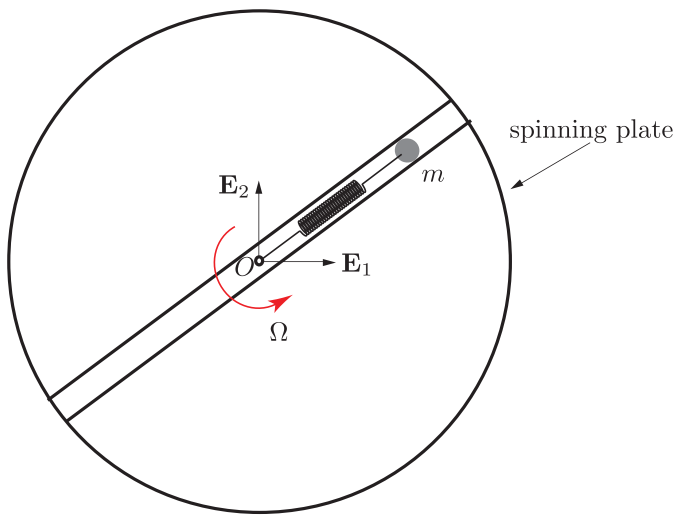


Figure 1: *Schematic of a particle of mass m which is attached to a fixed point O by an elastic spring. A vertical gravitational force $-mg\mathbf{E}_3$ acts on the particle and the particle is free to move in a smooth grove on a plate which is rotating about the vertical axis with a non-constant speed $\Omega = \Omega(t)$.*

In your answers to the questions below, please make use of the results on cylindrical polar coordinates on Page 3.

(a) What are the two constraints on the motion of the particle? Give prescriptions for the constraint forces enforcing these two constraints.

(b) Draw a freebody diagram of the particle. Your freebody diagram should include a clear expression for the spring force.

(c) Establish the second-order differential equation governing the motion of the particle.

(d) Show that the energy-like function

$$V = \frac{m}{2}\left(\dot{r}^2 - r^2\Omega^2\right) + \frac{K}{2}\left(r - \ell_0\right)^2 \tag{1}$$

is conserved provided Ω is constant.

Question 2

A Particle on a Smooth Surface

30 Points

As shown in Figure 2, a particle of mass m is free to move on a surface $z = f(x, y)$.

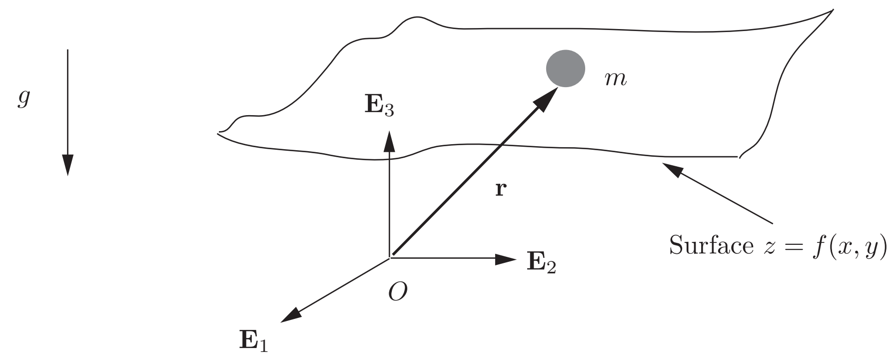


Figure 2: *Schematic of a particle of mass m which is moving on a smooth surface $z = f(x, y)$ in $\mathbb{E}^3$ under the influence of a gravitational force.*

To establish the equations of motion for the particle, the following curvilinear coordinate system is defined for $\mathbb{E}^3$:

$$q^1 = x, \qquad q^2 = y, \qquad q^3 = z - f(x, y). \tag{2}$$

(**a**) *(8 Points)* Show that the covariant basis vectors for this system are

$$\mathbf{a}_1 = \mathbf{E}_1 + \frac{\partial f}{\partial x}\mathbf{E}_3, \qquad \mathbf{a}_2 = \mathbf{E}_2 + \frac{\partial f}{\partial y}\mathbf{E}_3, \qquad \mathbf{a}_3 = \mathbf{E}_3. \tag{3}$$

Compute the matrix $[a_{ik}]$ and a. You may find it helpful to use the abbreviations $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.

(**b**) *(8 Points)* What are the contravariant basis vectors $\mathbf{a}^k$ for this coordinate system? Compute the inverse of the matrix $[a_{ik}]$.

(**c**) *(6 Points)* Assuming the particle is free to move on the smooth surface $z = f(x, y)$ under a gravitational force $-mg\mathbf{E}_3$, show that the equations of motion for the particle can be expressed in the form:

$$\frac{d}{dt}\left(?\dot{x}+??\dot{y}\right) - ??? = 0, \qquad \frac{d}{dt}\left(??\dot{x}+????\dot{y}\right) - ????? = 0, \tag{4}$$

For full credit supply the missing terms.

(**d**) *(4 Points)* Using the work-energy theorem, show that the total energy E of the particle is conserved.

(**e**) *(4 Points)* Show that the equations of motion of a particle moving on the smooth saddle $z = xy$ in the presence of a gravitational force $-mg\mathbf{E}_3$ can be expressed in the form

$$\mathsf{M}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \mathsf{f} = 0. \tag{5}$$

For full credit supply the components of the 2×2 mass matrix M and the 2×1 array f.

Notes on Cylindrical Polar Coordinates

Recall that the cylindrical polar coordinates $\{r, \theta, z\}$ are defined using Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$ by the relations:

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad z = x_3.$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

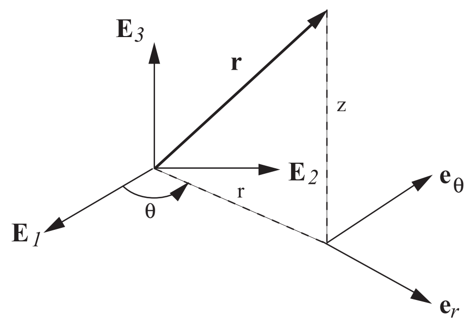


Figure 3: Cylindrical polar coordinates

For the coordinate system $\{r, \theta, z\}$, the covariant basis vectors are

$$\mathbf{a}_1 = \mathbf{e}_r, \quad \mathbf{a}_2 = r\mathbf{e}_\theta, \quad \mathbf{a}_3 = \mathbf{e}_z.$$

In addition, the contravariant basis vectors are

$$\mathbf{a}^1 = \mathbf{e}_r, \quad \mathbf{a}^2 = \frac{1}{r}\mathbf{e}_\theta, \quad \mathbf{a}^3 = \mathbf{e}_z.$$

The gradient of a function $u(r, \theta, z)$ has the representation

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{\partial u}{\partial \theta}\frac{1}{r}\mathbf{e}_\theta + \frac{\partial u}{\partial z}\mathbf{E}_3,$$

and the kinetic energy of a particle of mass m has the representation

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right). \tag{6}$$

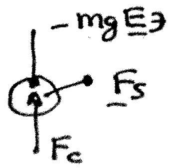
QUESTION 1

(a) Constraints

$$z = 0$$
$$\theta - f(t) = 0 \qquad \text{where} \qquad f(t) = \theta(t_0) + \int_{t_0}^{t} \Omega(\tau)\, d\tau$$

$$\underline{F}_c = \lambda_1 \underline{E}_3 + \frac{\lambda_2}{r} \underline{e}_\theta \qquad \text{(Lagrange's Prescription)}$$

(b)



$$\underline{F}_s = -K(r - \ell_0)\underline{e}_r$$

(c) Use Approach II

$$\tilde{T} = \tfrac{1}{2} m(\dot{r}^2 + r^2\Omega^2)$$

$$\tilde{U} = \tfrac{1}{2} K(r-\ell_0)^2 + mg(0) \quad , \quad \tilde{L} = \tilde{T} - \tilde{U})$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{r}} = m\dot{r}\right) - \left(\frac{\partial \tilde{L}}{\partial r} = mr\Omega^2 - K(r-\ell_0)\right) = 0$$

Equation of motion $\qquad m\ddot{r} - mr\Omega^2 + K(r-\ell_0) = 0.$

(d) To prove $\mathcal{V} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}K(r-\ell_0)^2 - \frac{1}{2}mr^2\Omega^2$ is conserved, we compute

$$\dot{\mathcal{V}} = m\dot{r}\ddot{r} + K(r-\ell_0)\dot{r} - mr\Omega^2\dot{r}$$
$$= \dot{r}(m\ddot{r} + K(r-\ell_0) - mr\Omega^2) = \dot{r}(0) = 0.$$

Hence $\mathcal{V}$ is conserved.

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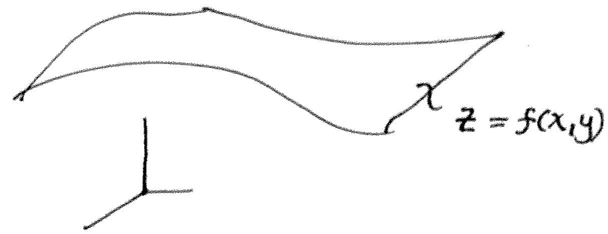
QUESTION 1



(a)

$$\underline{r} = x\,\underline{E}_1 + y\,\underline{E}_2 + (q^3 - f(x,y))\,\underline{E}_3$$

$q^1 = x$
$q^2 = y$
$q^3 = z - f(x,y)$

$$\underline{a}_1 = \frac{\partial \underline{r}}{\partial q^1} = \frac{\partial \underline{r}}{\partial x} = \underline{E}_1 + f_x\,\underline{E}_3$$

$$f_x = \frac{\partial f}{\partial x}$$

$$\underline{a}_2 = \frac{\partial \underline{r}}{\partial q^2} = \frac{\partial \underline{r}}{\partial y} = \underline{E}_2 + f_y\,\underline{E}_3$$

$$f_y = \frac{\partial f}{\partial y}$$

$$\underline{a}_3 = \frac{\partial \underline{r}}{\partial q^3} = \underline{E}_3$$

$$[a_{ik}] = \begin{bmatrix} 1 + f_x^2 & f_x f_y & f_x \\ f_x f_y & 1 + f_y^2 & f_y \\ f_x & f_y & 1 \end{bmatrix}$$

$$a = (\underline{a}_1 \times \underline{a}_2)\cdot \underline{a}_3 = (\underline{E}_3 - f_y\,\underline{E}_2 - f_x\,\underline{E}_1)\cdot \underline{E}_3 = \underline{1}$$

(b)

$$\underline{a}^1 = \operatorname{grad}(q^1) = \underline{E}_1$$

$$\underline{a}^2 = \operatorname{grad}(q^2) = \underline{E}_2$$

$$\underline{a}^3 = \operatorname{grad}(q^3) = \underline{E}_3 - f_x\,\underline{E}_1 - f_y\,\underline{E}_2$$

$$[a_{ik}]^{-1} = [a^{ik}] = \begin{bmatrix} 1 & 0 & -f_x \\ 0 & 1 & -f_y \\ -f_x & -f_y & 1 + f_x^2 + f_y^2 \end{bmatrix}$$

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(c) Particle is on smooth surface $z = f(x,y)$: $q^3 = 0$

$$\underline{N} = \underline{F}_c = \lambda \underline{a}^3$$

$$\tilde{T} = \frac{1}{2} m \left(\dot{x}^2(1+f_x^2) + \dot{y}^2(1+f_y^2) + 2 f_x f_y \dot{x}\dot{y} \right)$$

$$\tilde{U} = mgf$$

Lagrange's Equations (Approach II)

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}} = m(1+f_x^2)\dot{x} + f_x f_y \dot{y}\right) + mg f_x - \frac{\partial \tilde{T}}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{y}} = m(1+f_y^2)\dot{y} + f_x f_y \dot{x}\right) + mg f_y - \frac{\partial \tilde{T}}{\partial y} = 0$$

where

$$\frac{\partial \tilde{T}}{\partial x} = m f_x f_{xx} \dot{x}^2 + m f_y f_{yx} \dot{y}^2 + m f_{xx} f_y \dot{y}\dot{x} + m f_x f_{xy} \dot{y}\dot{x}$$

$$\frac{\partial \tilde{T}}{\partial y} = m f_x f_{xy} \dot{y}^2 + m f_y f_{yy} \dot{y}^2 + m f_{xy} f_y \dot{y}\dot{x} + m f_x f_{yy} \dot{y}\dot{x}$$

(d) From Work-Energy theorem

$$\dot{E} = \underline{F}_{nc} \cdot \underline{v} = \underline{N} \cdot \underline{v} = \lambda \underline{a}^3 \cdot \underline{v} = 0$$ because $\underline{v} = \dot{x}\underline{a}_1 + \dot{y}\underline{a}_2$

Hence E is conserved where $E = \tilde{T} + \tilde{U}$.

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(e) $f = xy$

$$\tilde{T} = \tfrac{1}{2} m(1+y^2)\dot{x}^2 + \tfrac{1}{2} m(1+x^2)\dot{y}^2 + mxy\dot{x}\dot{y}$$

$$\tilde{U} = mgxy$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{x}} = m(1+y^2)\dot{x} + mxy\dot{y}\right) - \left(\frac{\partial \tilde{T}}{\partial x} = mx\dot{y}^2 + my\dot{x}\dot{y}\right)$$

$$= -mgy$$

$$\frac{d}{dt}\left(\frac{\partial \tilde{T}}{\partial \dot{y}} = m(1+x^2)\dot{y} + mxy\dot{x}\right) - \left(\frac{\partial \tilde{T}}{\partial y} = my\dot{x}^2 + mx\dot{x}\dot{y}\right)$$

$$= -mgx$$

Expanding $\frac{d}{dt}$ and collecting terms

$$\Rightarrow \begin{bmatrix} m(1+y^2) & mxy \\ mxy & m(1+x^2) \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 2my\dot{y}\dot{x} + mgy \\ 2mx\dot{y}\dot{x} + mgx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$