Midterm 2 - Solutions

1. [25 points] Short Questions

(a) [5 points] Circle (black in) T or F for True or False

T (i) Unlike electric charges attract and like charges repel.

T (ii) The net amount of charge produced in any process is zero.

F (iii) The electric field intensity anywhere is the sum of the electric field intensity from all the contributing charges.

answer: The field strength IS the sum, but the field INTENSITY is not.

T (iv) Electric charge is quantized.

T F (v) The electric potential of a system of two unlike charges is positive.

NOTE: This question is ambiguous so it was not graded.

F (vi) If there is no charge in a region of space, the electric field on a surface surrounding the region must be zero everywhere.

answer: the FLUX must be zero everywhere, but not the FIELD.

T (vii) In electrostatic equilibrium, the electric field inside a conductor is zero.

F (viii) If the net charge on a conductor is zero, the charge density must be zero at every point on the surface of the conductor.

answer: imagine a conducting shell with +Q on its outer surface and -Q on its inner surface.

T (ix) The result that $E = 0$ inside a conductor can be derived from Gauss's Law.

F (x) One can induce charge on a conductor without touching.

F (xi) Resistance increases as temperature increases.

(b) [5 points] Circle (black in letter of) correct answer

(i) Metals are in general electrical
(A) nonconductors.
(B) conductors.
(C) semiconductors
(D) insulators..
(E) conductance depends upon configuration.
Answer: (B)

(ii) What is the maximum voltage that a sphere of radius 5 cm can hold in air?
(A) 3×10^6 V.
(B) 1.5×10^6 V.
(C) 3×10^5 V.
(D) 1.5×10^5 V.
(E) 3×10^4 V.
Answer: (D) 1.5×10^5 V.
Reason: Air breaks down at an electric field of 3×10^6 V/m. The electric field is $E = Q/(4\pi\epsilon_0 r^2) = 3 \times 10^6$ V/m or $Q = 4\pi\epsilon_0 r^2 E = 4\pi \times 8.85 \times 10^{-12} \times 0.05^2 \times 3 \times 10^6 = 8.34 \times 10^{-7}$ coulombs. Voltage is given by $V = Er = 3 \times 10^6 \times 0.05 = 1.5 \times 10^5 V$.

(iii) How much voltage is necessary to accelerate a proton so that it has just sufficient energy to touch the surface of an iron nucleus? An iron nucleus has a charge of 26 times that of a proton (e) and its radius is about 4.0×10^{-15}m. Whereas the proton has a radius of about 1.2×10^{-15}m. Assume the nucleus is spherical and uniformly charged.
(A) 13.6 V.
(B) 354 V
(C) 7000 V.
(D) 500,000 V.
(E) 7,000,000 V.
Answer: (E) 7,000,000 V.
Reason: Potential Energy is $eV = \frac{e26e}{4\pi\epsilon_0 d^2} = 26 \times 10^{-19} e/(4\pi \times 8.85 \times 10^{-12} \times)(1.2 + 4.0) \times 10^{-15} eV = 7.23(9.4\ if\ use\ iron\ radius) \times 10^6 eV$

(iv) A current of 10 amps in a 2-mm diameter copper wire is the result of an electron drift velocity of about
(A) 2.5×10^{-6} m/s
(B) 2.5×10^{-4} m/s
(C) 2.5×10^{-2} m/s
(D) 2.5×10^{0} m/s
(E) 2.5×10^{2} m/s
Answer: (B) 2.5×10^{-4} m/s
Reason: $I = \rho v \pi r^2, so v = \frac{I}{\rho \pi r^2} = \frac{10A}{9 \times 10^{28} m^{-3} \times 1.6 \times 10^{-19} \pi (0.001)^2}$

(v) The electric flux from a cubical box 28 cm on a side immersed in water (dielectric constant = 80) is 1.45 $\times 10^3 N \cdot m^2/C$ for a Gaussian surface in the water. What is the net charge enclosed in the box?

(A) 1 pC
(B) 1 nC
(C) 1 μC
(D) 1 mC
(E) 1 C.

Answer: (C) 1 μC

Reasoning: Use Gauss's Law $\Phi_E = q_{net}/\epsilon = q_{net}/(K\epsilon_0)$ $q_{net} = 1.45 \times 10^3 \times 8.85 \times 10^{-12} \times 80 = 1.027 \times 10^{-6} C$

(c) [5 points] The figure just below supposedly shows the electric field lines near an irregular-shaped positively-charged conductor. There are five distinctly different errors in the figure. Make a list of errors and give a brief reason why each is an error.
Multiple instances of the same error are **not** different.

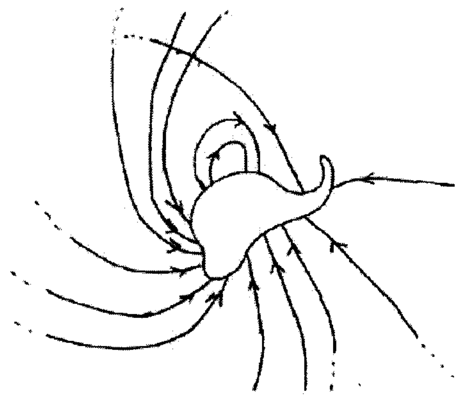


Figure 2: Figure for problem 1(c) showing field lines and conductor

(1) Field lines leave a positive charge. The field lines should all leave the conductor.

(2) The conductor has constant potential everywhere. Thus field lines cannot go from the surface to the same surface.

(3) Field lines cannot cross.

(4) There should be more field lines where the surface is more convex (smaller radius) compared to where the surface is less convex (larger radius).

(5) Field lines should be normal to the surface of the conductor.

(d) [5 points] Circle correct answer

(i) At a point high in the atmosphere, He^{++} ions in a concentration of $2.8 \times 10^{12}/m^3$ move north with a speed of $2 \times 10^6 m/s$. The current density at that point is

(A) $5.6 \times 10^{18} A/m^2$ north
(B) $11.2 \times 10^{18} A/m^2$ north
(C) $1.8 A/m^2$ north
(D) $3.6 A/m^2$ north
(E) none of the above as it is going south
Answer: (C) $1.8 A/m^2$ north
Reason: $\vec{j} = nq\vec{v} = 2.8 \times 10^{12}/m^3 \times 2 \times 1.6 \times 10^{-19} C \times 2 \times 10^6 m/sec = 1.79 C/m^2 \cdot sec$

(ii) A proximity (don't have to touch only get very near) button makes use of what effect?

(A) Ease of sensing the residual electric field on human body.
(B) Heat sensing due to the blood in the finger tips..
(C) Sensing of the electrical currents in finger tips.
(D) Sensing the potential difference between the body and the button.
(E) Capacitance, which changes as a conducting finger is brought near the button.
Answer: (E)

(iii) The wiring in a house must be thick enough so it does not become so hot as to start a fire. What diameter must a copper wire, $\rho = 1.68 \times 10^{-8} \Omega \cdot m$, if it is to carry a maximum current of 20 A and produce no more than 2 W of heat per meter?

(A) 0.5 mm.
(B) 1 mm.
(C) 1.5 mm.
(D) 2 mm.
(E) 2.5 mm.
Answer: (D) 2 mm.
Reason: $R = \rho L/A = 1.68 \times 10^{-8} \Omega \cdot m \times 1m/(\pi d^2/4)$ $P = I^2 R = 20^2 \times 1.68 \times 10^{-8} \Omega \cdot m^2/(\pi d^2/4) = 2w$ $d = \sqrt{4 \times 20^2 \times 1.68 \times 10^{-8} \Omega \cdot m^2/2w\pi} = 2.068 \times 10^{-4} m = 2.068 mm$

(iv) At what current do humans experience severe heart fibrillation most lethally?

(A)1 mA.
(B) 10 mA.
(C) 100 mA.
(D) 1 A.
(E) 1 μA.
Answer: (C) 100 mA

(v) A bird perches on a DC electric transmission line carrying 2500 A. The line has a resistance of $2.5 \times 10^{-5} \Omega$ per meter and the bird's feet are 4 cm apart. What potential difference does the bird feel?

(A) 2.5 μV.
(B) 2.5 mV.
(C) 2.5 V.

(D) 25 V.
(E) 250 V.
Answer: (B) 2.5 mV.
$V = RI = 0.04 \times 2.5 \times 10^{-5}\Omega \times 2500A = 2.5mV$

(e) [5 points] Circle correct answer

(i) Four 100 Ω resistors are connected in parallel. What is the equivalent resistance?
(A) 25 Ω.
(B) 100 Ω.
(C) 400 Ω.
(D) 10 Ω.
(E) none of the above.
Answer: (A) 25 Ω.
Reason: $R_{equ} = (\frac{4}{100\Omega})^{-1}$

(ii) Four 100 μF capacitors are connected in parallel. What is the equivalent capacitance?
(A) 25 μF.
(B) 100 μF.
(C) 400 μF.
(D) 10 μF.
(E) none of the above.
Answer: (C) 400 μF.
Reason: $C_{equ} = 4 \times 100\mu F$

(iii) A 1200 watt hair dryer runs on 120 V AC power in your house. What is the peak current through the hair dryer?
(A) 1 A.
(B) 1.4 A.
(C) 10 A.
(D) 14 A.
(E) none of the above
Answer: (D) 14 A
Reason: $P = (1/2)I_0V_0, soI_0 = \frac{2P}{v0} = \frac{2\times1200W}{\sqrt{2}\times120V}$

(iv) The EV-1 electric car is powered by 26 batteries, each 12 V and 52 A-hr. Assume the car is on the level moving at 40 km/hr so that has an average retarding force of 240 N. After how many kilometers must the batteries be recharged?
(A) 2.5 km
(B) 25 km.
(C) 250 km.
(D) 2500 km.
(E) none of the above.
Answer: (C) 250 km.
$P = Fv = 240N \times 40km/hr = 2666.7w$ $Work = Fd = 240N \times d = 26 \times 12V \times 52A - hr = 16224W - hr = 5.84 \times 10^7 joules$ $d = W/F = 5.84 \times 10^7 joules/240N = 243 \times 10^3 m = 243km$

(v) A capacitor is often used in electronics to keep energy powering a circuit even if there is a momentary loss of power from the electric company. What capacitance would be required for a TV, operating at an internal voltage of 120 V at 150 W to provide sufficient energy during a 0.1 second lapse in power?
(A) 2 pF = 2 $\times 10^{-12}$ F

(B) 2 nF = 2×10^{-9} F
(C) 2 μF = 2×10^{-6} F
(D) 2 mF = 2000 μF = 2×10^{-3} F
(E) 12 F
(F) 2000 F

Answer: (D) 2 mF = 2000 μF = 2×10^{-3} F
Reasoning: $E = P\Delta T = 150W \times 0.1sec = 15J = \frac{1}{2}CV^2 = \frac{1}{2}120^2 C$
$C = 2E/V^2 = 2 \times 15J/120^2 = 2.083 \times 10^{-3} F$

2. [15 points] A touted breakthrough in high fidelity (Hi-Fi) was the development capacitance speakers. There are large parallel plates (1.5 m by 0.3 m), one of which is fixed in a fairly rigid frame and the other separated by 2mm of a thin soft foam with dielectric constant $\epsilon = 1.2\epsilon_0$. When the plates are charged, they attract, compressing the foam.
(a) [4 points] How much charge and electrical energy is stored in the capacitor at 600 volts if the plates did not get closer together?

The area of the plates is $A = 1.5m \times .3m = .45 m^2$

For a parallel plate capacitor, $C = \frac{A\epsilon}{\ell}$.

The charge stored in a capacitor at voltage V is $Q = CV$. Thus,

$$Q = \frac{A\epsilon}{\ell} V = \frac{(.45m^2)\,1.2\epsilon_0}{.002\,m} 600V = \boxed{1.43\times10^{-6} C}$$

The energy stored is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{A\epsilon}{\ell}V^2 = \frac{1}{2}\frac{(.45m^2)(1.2\epsilon_0)}{.002\,m}(600V)^2 = \boxed{4.30\times10^{-4} J}$$

(b) [6 points] When a voltage $V = 600$ Volts is applied across the plates how much does the separation of the plates change, if the spring constant of the foam is $k = 10^3 N/m$? What is the force on the plate? How much energy is stored in the compression and how does it compare to the stored electrical energy?

Since all the charge on one plate is seeing the same electric field, we can write

$$F_{\text{on plate}} = QE = Q\left[\frac{\sigma}{2\epsilon}\right] = \frac{Q^2}{2\epsilon A} = \frac{(1.43\times10^{-6}C)^2}{2.4\epsilon_0(.45m^2)} = \boxed{.215 N}$$

The ~~el~~ spring force must balance this force, so

$$F_{\text{spring}} = -k\Delta\ell = -F_{\text{on plate}} \Longrightarrow \Delta\ell = \frac{.215N}{10^3 N/m} = \boxed{2.15\times10^{-4} m}$$

The stored energy in the foam is $U = \frac{1}{2}k(\Delta\ell)^2 = \frac{1}{2}(10^3 N/m)(2.15\times10^{-4}m)^2 = \boxed{2.31\times10^{-5} J}$

At this new separation, $\ell = .002m - \Delta\ell = .001785\,m$, the energy stored becomes $U = \frac{1}{2}\frac{(.45m^2)(1.2\epsilon_0)}{.001785m}(600V)^2 = \boxed{4.82\times10^{-4} J}$ which is larger by 20x.

(c) [5 points] If the audio signal is at a frequency 10 kHz, how much electrical power goes to the speakers? How much goes to sound = moving air and the speaker plates?

The voltage swings from $-600V$ to $+600V$ at a frequency of $10^4 Hz$. This is a peak to peak AC voltage, so the power is given by

$$P = \frac{1}{2}U_{max} f = \frac{1}{2}(4.82\times10^{-4} J)10^4 s^{-1} = \boxed{2.41 \text{ watts}}$$

Similarly, the power to the moving air will be half the maximum energy stored in the foam. Thus,

$$P = \frac{1}{2}(2.31\times10^{-5} J)10^4 s^{-1} = \boxed{.115 \text{ watts}}$$

3. [15 points] The HCl molecule has a dipole moment of about $3.4 \times 10^{-30} C \cdot m$. The two atoms are separated by about $1.0 \times 10^{-10} m$.
(a) [4 points] What is the net charge on each atom?

$$\vec{p} = q\vec{d}$$

$$q = \frac{p}{d} = \frac{3.4 \times 10^{-30}}{1 \times 10^{-10}} = 3.4 \times 10^{-20}.$$

(b) [3 points] Is this equal to an integer multiple of e? If not, explain.

no, electron shared unequally between the H & Cl cores.

(c) [4 points] What maximum torque would this dipole experience in a $2.5 \times 10^4 N/C$ electric field?

$$\vec{\tau} = \vec{p} \times \vec{E}$$

$$|\vec{\tau}| = pE\sin\theta$$

$$\tau_{max} = pE = 3.4 \times 10^{-30} \times 2.5 \times 10^4$$

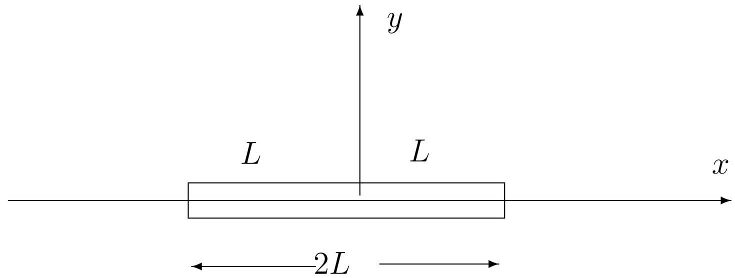
$$= 8.5 \times 10^{-26}\ N \cdot m$$

(d) [4 points] How much energy would be needed to rotate one molecule 45° from its equilibrium position of lowest potential energy?

$$U = -\vec{p} \cdot \vec{E} = -pE\cos\theta$$

$$\Delta U = U_f - U_i = -pE\left(\cos\left(\frac{\pi}{4}\right) - \cos(0)\right) = 2.49 \times 10^{-26}\ J$$

4. [20 points] A thin rod of length $2L$ is centered on and lying on the x axis. The rod carries



a uniformly distributed charge Q.

(a) [5 points] Determine the potential $V(y)$ as a function of y (perpendicular direction to rod) for points on the y-axis. Let $V = 0$ at infinity.

Answer: $dV = dQ/4\pi\epsilon_0 r$ where $dQ = Qdx/(2L)$ and $r^2 = x^2 + y^2$ or $r = \sqrt{x^2+y^2}$
$V(y) = \frac{Q}{4\pi\epsilon_0 2L}\int_{-L}^{L}\frac{dx}{\sqrt{x^2+y^2}} = \frac{Q}{4\pi\epsilon_0 2L}arcsinh(x/y)|_{-L}^{L} = \frac{Q}{4\pi\epsilon_0 2L}(arsinh(L/y) - arcsinh(-L/y))$
which also equals $= \frac{Q}{4\pi\epsilon_0 2L}log(\frac{\sqrt{L^2+y^2}+L}{\sqrt{L^2+y^2}-L})$

(b) [5 points] Determine the potential $V(x)$ for points along the x axis outside the rod.

Answer: $dV = dQ/4\pi\epsilon_0 r$ where $dQ = Qdx/(2L)$ and $r^2 = (x'-x)^2+y^2$ or $r = \sqrt{(x'-x)^2+y^2}$ and $y = 0$ $V(y) = \frac{Q}{4\pi\epsilon_0 2L}\int_{-L}^{L}\frac{dx}{x'-x} = \frac{Q}{4\pi\epsilon_0 2L}ln(x'-x)|_{-l}^{L} = \frac{Q}{4\pi\epsilon_0 2L}ln(x+L)/(x-L))$

(c) [5 points] What is the electric field along the y axis?

Answer: We can derive this either of two ways:

(1) $\vec{E} = -\nabla V(y) = \int_{-L}^{L}\frac{Qy}{4\pi\epsilon_0 2L(x^2+y^2)^{3/2}}dx\hat{y}$

(2) Integrate $d\vec{E} = \frac{dQ\hat{r}}{4\pi\epsilon_0 r^2} = \frac{dQ\hat{r}}{4\pi\epsilon_0(x^2+y^2)} = \frac{Q}{4\pi\epsilon_0 2L}\int\frac{dx}{x^2+y^2}\hat{r} = \frac{Q}{4\pi\epsilon_0 2L}\int\frac{ydx}{(x^2+y^2)^{3/2}}\hat{y}$ since only the *haty* direction adds coherently the $\hat{x}$ has canceling components and there is no *hatz* component along the y direction by symmetry.

$\vec{E} = \frac{Q}{4\pi\epsilon_0 2L}\frac{yx}{y^2\sqrt{x^2+y^2}}|_{-L}^{L} = \frac{Q}{4\pi\epsilon_0 2L}\left(\frac{L}{y\sqrt{y^2+L^2}} - \frac{-L}{y\sqrt{y^2+L^2}}\right) = \frac{Q}{4\pi\epsilon_0}\frac{1}{y\sqrt{y^2+L^2}}$

Note that this has the correct dependence: it goes down as $1/r$ near ($d < L$) the rod and as $1/r^2$ far from the rod.

(d) [5 points] Place a sphere of radius R at $y = d$, $x = 0$ with $d < R < L$. What is the electric flux through the sphere?

Answer: The easiest way to work this problem is to use Gauss's law $\Phi_E = Q_{enclosed}/\epsilon_0$ We can find the enclosed charge by multiplying the linear charge density times the length of the rod enclosed by the sphere. By simple trigonometry $\ell = \sqrt{R^2-d^2}$ is half the length in the sphere. $Q_{enclosed} = \lambda(2\ell) = \frac{Q}{2L}2\sqrt{R^2-d^2} = Q\sqrt{R^2-d^2}/L$ Thus the flux is

$\Phi_E = Q_{enclosed}/\epsilon_0 = \frac{Q}{\epsilon_0}\frac{\sqrt{R^2-d^2}}{L}$

Problem 5 solutions

a. This is a simple circuit consisting of the 110 V source and the person's resistance.
$I = V/R = 110/1100 = 0.1$A

b. We have two resistors in parallel now, but each still sees the same total voltage, thus the body receives the same current, 0.1A. The alternate path receives $110/40 = 2.75$A, and so the voltage source is supplying a total of 2.85A. However, this is irrelevant; the person sees only 0.1A (100mA).

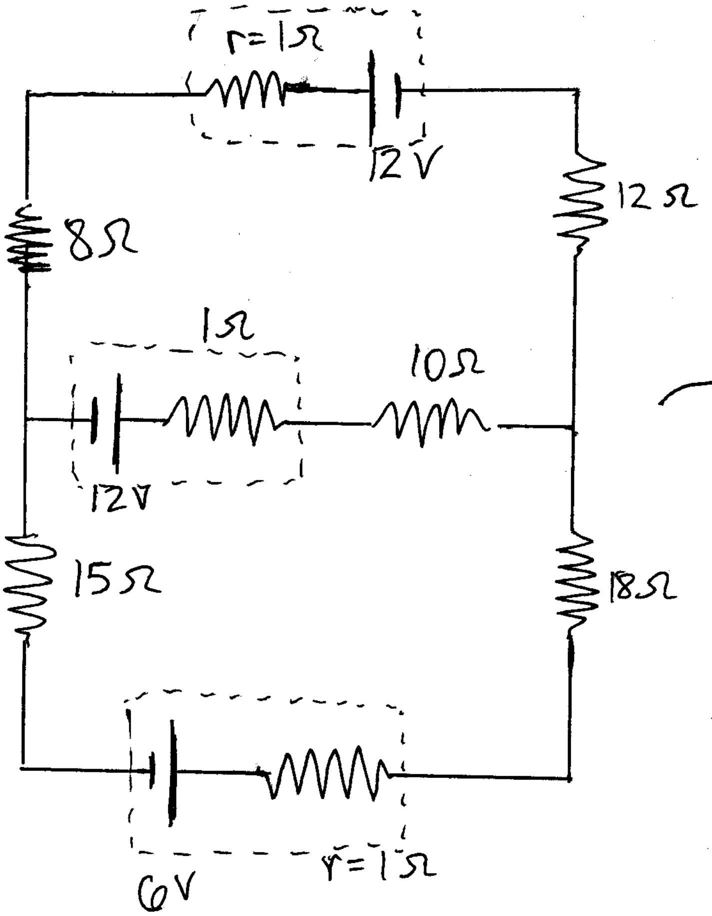
c. Now if the voltage source can supply at most 1.5A, then we have to worry about the other resistor. The equivalent single resistor corresponding to the person and alternate path in parallel must have a resistance of 38.6Ω. The voltage source therefore has an effective maximum voltage of $1.5 \times 38.6 = 57.9$V. Thus the person sees a current of $57.9/1100 = 52.6$mA. Alternatively, we may realize that the total current in the case of a perfect EMF was 2.85A, and so we must scale all the currents we actually see down by a factor of $1.5/2.85 = 0.526$, which turns 100mA into 52.6mA.

d. Heart fibrillation requires 100mA. In part b, we will see this amount and so will be in danger. In part c, we see only 52.6mA and so we're safe.

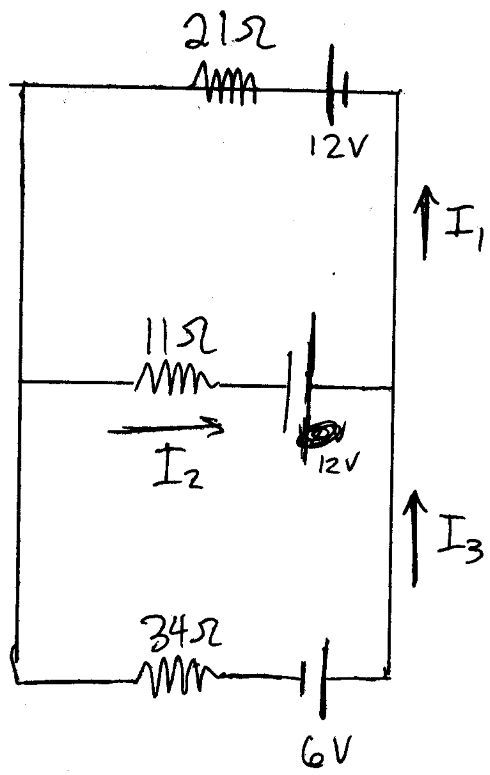
e. The body's resistance is 1100Ω. If we are to supply 1A across a person, the voltage source must be 1100V. Recall the energy stored in a capacitor is given by $U = CV^2/2$. Since we know $U = 200$J and $V = 1100$V we can calculate C, giving $C = 331\mu$F.

6 Solutions (21 points total)

(a)



Ans:



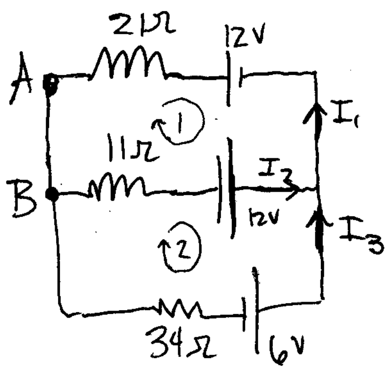
- We want the minimum number of resistors so we combine all the resistors in series into one resistor on each branch.
- We can't say the resistors are in parallel because the batteries mess up our assumptions about voltage drops on each branch.

Grading: + 2 pts for correct top branch
+1 pt for correct middle
+2 pts for correct bottom

(5 points)

(b) Solve for I_1, I_2, I_3:

Ans:



We have 3 unknowns, so we need 3 equations.

Kirchoff Junction rule:

$$I_1 = I_2 + I_3$$

6b - cont'd

Now, we walk around the circuit adding V drops and rises.

Loop/Path 1: start at upper left (node A), go clockwise

$$21 I_1 - 12 - 12 + 11 I_2 = 0$$

Loop/Path 2: Start at node B, go clockwise

$$-11 I_2 + 12 - 6 + 34 I_3 = 0$$

Collect all 3 equations and rearrange:

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ 21 I_1 + 11 I_2 &= 24 \\ 11 I_2 - 34 I_3 &= 6 \end{aligned}$$

Solving this system is just algebra, and it's not too bad. One method is to use matrices:

$$\underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 21 & 11 & 0 \\ 0 & 11 & -34 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 \\ 24 \\ 6 \end{pmatrix}}_{\vec{b}} \Rightarrow A^{-1} \vec{b} = \vec{x}$$

(inverse of A

$$\Rightarrow \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 1014/1319 \text{ A} \\ 942/1319 \text{ A} \\ 72/1319 \text{ A} \end{pmatrix} = \boxed{\begin{pmatrix} 0.77 \text{ A} \\ 0.71 \text{ A} \\ 0.055 \text{ A} \end{pmatrix}}$$

Grading: +1 pt. for recognizing need for 3 equations
+1 pt for each correct loop rule (×2)
+1 pt for correct answer

5 pt.

6-ⓒ What is the terminal voltage on the 6 V battery?

Ans: $V_{terminal} = \mathcal{E} - iR$

$= (6V) - I_3 r$

$= 6V - (0.055A)(1\,\Omega)$

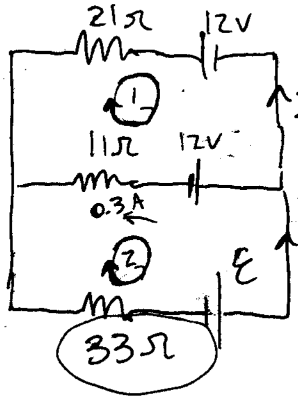
$= 5.94\ V$

Grading: +1 for using $V = \mathcal{E} - ir$
+1 for using correct R
+1 for using I_3

Note: I did not dock points if your answer to (b) was incorrect but you stayed consistent.

ⓓ Replace the 6 V battery with an unknown emf, $\mathcal{E}$. If $I_2 = -0.3\ A$, what is the $\mathcal{E}$?

Our equations from part (b) become:



$I_1 = I_3 - 0.3$

$21 I_1 - 12 - 12 + 11(-0.3) = 0$

$-11(-0.3) + 12 - \mathcal{E} + 33 I_3 = 0$

$\Downarrow$

$I_1 - I_3 = -0.3$

$21 I_1 = 27.3$

$\mathcal{E} \quad - 33 I_3 = 15.3$

$I_1 = 1.3\ A$

$\Rightarrow I_3 = 1.6\ A$

$\Rightarrow \mathcal{E} = 15.3 + 33 \cdot 1.6$

$= \boxed{68\ V}$

Note: We replaced the battery ($\mathcal{E}$ + internal resistance) by an emf. Thus, the r is gone

Grading: +1 pt for each correct equation (×3)
+1 pt for answer
4 pts

I did not deduct points if you left the internal resistance. In that case $\mathcal{E} = 69.7\ V$. The problem was not totally clear on that.

6-(e) What would be the current if the 18 Ω resistor were shorted out?

<u>Ans</u>: Short = replaced with a wire $\Rightarrow$ act as if 18 Ω resistor is gone

Thus, equations from part (a) just become ($34\Omega \to 16\Omega$)

$$I_1 - I_2 - I_3 = 0$$
$$21I_1 + 11I_2 = 24 \quad \Rightarrow \quad \boxed{I_1 = 0.78\text{ A}}$$
$$11I_2 - 16I_3 = 6$$

$$I_2 = 0.69\text{A}$$
$$I_3 = 0.097\text{ A}$$

<u>Grading</u>: +1 for each correct equation (x3)
+1 for answer
4 points

<u>Special case</u>: if you thought that a "short" meant that there was a break then I gave <u>1 pt</u> if you followed through correctly.

General Comments:

- I realize that you basically solve the same problem 3 times. Unfortunately, I could only give credit for work that you show even if you seem to demonstrate an understanding of how to solve the problem. It broke my heart, but that's life.
- You can use a calculator to solve; the actual solution (solving the 3 linear equations) was worth 0 points. anyway.

Solution

(a) We can find the potential of the inner sphere by first finding the electric field using Gauss's law. Using a concentric sphere of radius r for our Gaussian surface and taking into account the constraint that $\vec{E} = 0$ in the conductor gives us the result that in spherical coordinates with the origin at the center of the spheres:

$$\vec{E}(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}, & R_3 < r \text{ or } R_1 < r < R_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The potential at the surface of the inner sphere with respect to infinity is now found from:

$$V(R_1) = -\int_{\infty}^{R_1} \vec{E} \cdot d\vec{r} \tag{2}$$

Since the electric field is defined in a peicewise manner, we need to break this integral up into parts.

$$\begin{aligned} V(R_1) &= -\int_{\infty}^{R_1} \vec{E} \cdot d\vec{r} \\ &= -\int_{\infty}^{R_3} \vec{E} \cdot d\vec{r} - \int_{R_3}^{R_2} \vec{E} \cdot d\vec{r} - \int_{R_2}^{R_1} \vec{E} \cdot d\vec{r} \\ &= -\int_{\infty}^{R_3} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr + 0 - \int_{R_2}^{R_1} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr \qquad (3) \\ &= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R_3} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (4) \end{aligned}$$

(b) Now with the outside shell grounded, the potential difference between the outside shell and infinity is zero, so the electric field outside the shell is zero. Our field is now:

$$\vec{E}(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}, & R_1 < r < R_2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Our calculation for the potential of the inner sphere is the same, except now the first integral in (3) vanishes and we have:

$$V(R_1) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{6}$$

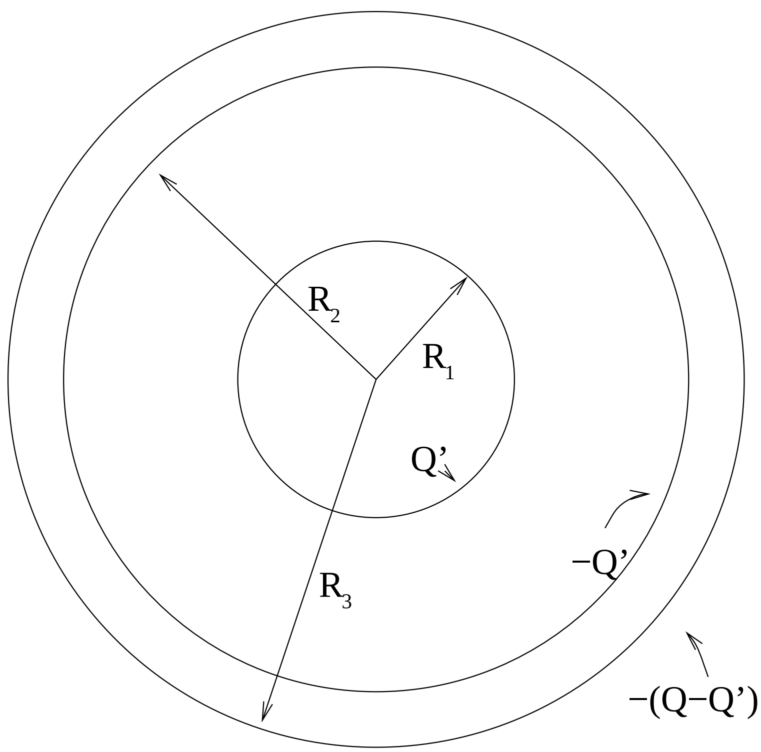
(c) In order to achieve the fields in (5), the outer conducting shell must aquire a charge $-Q$. Now the outer shell has been disconnected, so it is stuck with this *net* charge of -Q on it, regardless of what we do with the inner shell. Grounding out the inside sphere sets the condition

$$V(R_1) = V(\infty) \tag{7}$$

We can now see that this condition implies that there must be charge on the inner sphere, because we can rewrite (7) as

$$[V(R_1) - V(R_2)] - [V(\infty) - V(R_2)] = 0 \tag{8}$$

Because there is a net negative charge on the outer conductor, we expect the second term in brackets to be non-zero[1] , and therefore in order to satisfy (8), we must have the first term in brackets non-zero as well. This means there must be some electric field between the plates, which implies that the charge on the inner conductor is not zero. Let's assume that a charge Q' accumulates on the inner sphere, and solve for this charge Q'. To satisfy the conditionthat the electric field be zero in the conducting shell, our picture must look like:



[1]You can easily verify that it is impossible to adjust the charge on the inner conductor so that (8) is satisfied by making both terms zero.

Now the electric field is found using Gauss's law to be:

$$\vec{E}(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q'}{r^2}\hat{r}, & R_1 < r < R_2 \\ \frac{1}{4\pi\varepsilon_0}\frac{-(Q-Q')}{r^2}\hat{r}, & r > R_3 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Now that we have the field, the condition of (7) becomes:

$$\begin{aligned} -\int_{\infty}^{R_1} \vec{E}\cdot d\vec{r} &= -\int_{\infty}^{R_3} dr \frac{1}{4\pi\varepsilon_0}\frac{-(Q-Q')}{r^2} + 0 - \int_{R_2}^{R_1} dr \frac{1}{4\pi\varepsilon_0}\frac{Q'}{r^2} \\ 0 &= \frac{1}{4\pi\varepsilon_0}\frac{-(Q-Q')}{R_3} + \frac{Q'}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \end{aligned} \tag{10}$$

Now we can solve the above equation for Q' with the result:

$$Q' = \frac{Q}{R_3}\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3}\right]^{-1} \tag{11}$$

Then recognizing that the first term in (10) is the potential of the outer shell with respect to infinity, we have that:

$$V(R_3) = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{R_3}\left(1 - \frac{1}{R_3}\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3}\right]^{-1}\right) \tag{12}$$

(d) This is a standard capacitance problem.We can figure out the capacitance that would occur with vacuum in between the plates, and then multiply this capacitance by k. Since in part (b) the outer sphere is grounded, the potential between R_1 and infinity is the same as the potential between R_1 and R_2, so we can use (6) for the potential difference created by putting a charge $-Q$ on the outer shell and $+Q$ on the inner sphere. The vacuum capacitance is then found from:

$$C_{vac} = \frac{Q}{V_{vac}} = 4\pi\varepsilon_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^{-1} \tag{13}$$

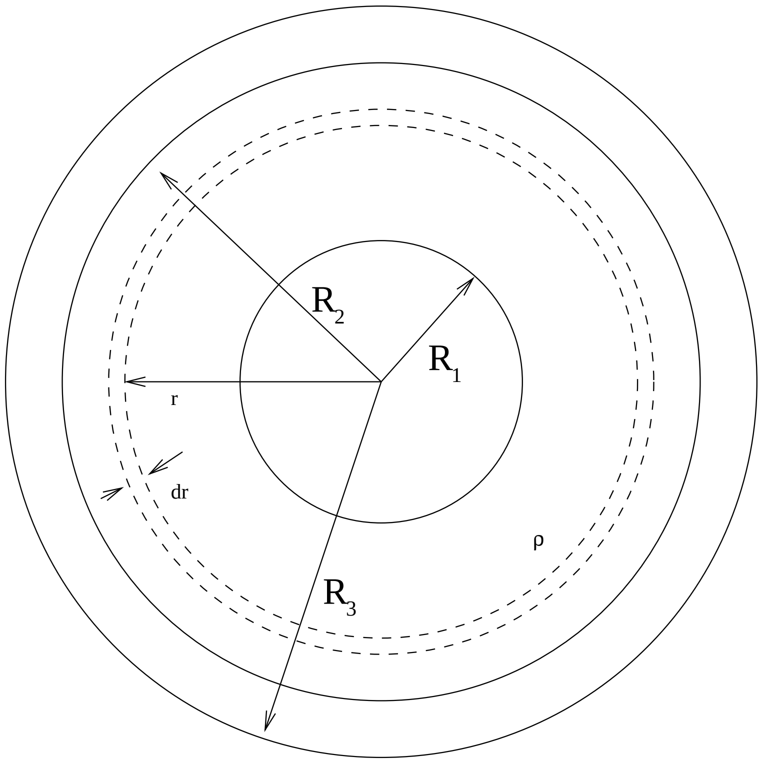
So with the water in between the plates:

$$C = kC_{vac} = 4\pi k\varepsilon_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^{-1} = 320\pi\epsilon_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^{-1} \tag{14}$$

(e) Now we would like to use the formula

$$R = \frac{\rho l}{A}, \tag{15}$$

but we can't because this formula only works for simple slabs of material where the current goes straight through the slab without diverging out. Here we do not have a simple slab. But we can imagine making the resistor out of a series of infintesimal slabs. One of these is shown as a dotted ring below:



In the limit as $dr \to 0$, this looks like a slab of area $A = 4\pi r^2$ and length dr, so its differential resistance is:

$$dR = \rho \frac{dr}{4\pi r^2} \tag{16}$$

Adding up the resistance of all these rings in series, we have:

$$R = \int dR = \frac{\rho}{4\pi} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{\rho}{4\pi} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \tag{17}$$

If you don't like this method, another way to do the problem is to look at the flux of the $\vec{J}$ vector, which is given by Ohm's law as:

$$\vec{J} = \frac{1}{\rho} \vec{E} \tag{18}$$

Now $\left|\vec{J}\right|$ is the current per unit area, so the flux of $\vec{J}$ through an imagnary sphere concentric with the inner sphere will be the total current I:

$$I = \oint \vec{J} \cdot d\vec{a} = \frac{1}{\rho} \oint \vec{E} \cdot d\vec{a} \tag{19}$$

But the flux integral of $\vec{E}$ we can identify with Gauss's law (modified in the presence of the dielectric $\varepsilon_0 \to k\varepsilon_0$):

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{k\varepsilon_0} \tag{20}$$

Now Ohm's law also says $R = V/I$ or using (6) again (modified for the dielectric):

$$R = \frac{V}{I} = \frac{\frac{Q}{4\pi k\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}{\frac{Q}{\rho k\varepsilon_0}} = \frac{\rho}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{21}$$

which is the same result as before.

Grading Scheme

Common mistakes and the points awardered are listed below. In general, half a point was deducted from answers that had the wrong sign where the sign of the answer has physical meaning and could be figured out from inspection of the proble. At least a full point was deducted from answers that had the wrong units.

Part (a)

- +2 points for
$$V(R_1) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
which is a good try, but doesn't take into account the fact that the potential of the outside shell with respect to infinity is not zero.

- +1 for
$$V(R_1) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R_1},$$
which would be true if the outside shell were absent.

- +2 or +3 for correctly finding the $\vec{E}$-field and having some reasonable scheme for doing the line integrals, but making some silly mistakes.

Part (b)

- +1 for saying that the potential was the same as found in part (a)

- +0 for saying that the potential was zero with no justification.

- +1 for saying that the potential was zero with some reasonable attempt at justification.
- +1 for
$$V(R_1) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R_1},$$
- +1.5 for calculating
$$V(R_1) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right),$$
which fails to take into account that the electric field is zero in the outer shell.

Part (c)

- +1.5 for saying that all the charge has been discharged and that the electric field and voltage are zero everywhere. This is a natural response to this question, and shows some understanding of how conductors accumulate/lose charge.
- +1 for saying zero with no justification.
- +2 for realizing that the negative charge on the outside shell is still there and saying:
$$V(R_3) = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{R_3}$$

Part (d)

- +1 for just pulling an answer out of thin air with absolutely no justification.
- +2 or +3 for finding the capacitance in the standard way but making some silly mistakes.
- +2 for writing down a sensible wrong answer based on previous calculations in the problem.
- +2 for obtaining
$$C = 4\pi k\varepsilon_0\left(\frac{1}{R_1} - \frac{1}{R_3}\right)^{-1}$$
after doing the line integral in from R_3.

Part (e)

- +1 for an approximation like:

$$R = \frac{\rho l}{A} \approx \frac{\rho(R_2 - R_1)}{4\pi R_2^2}$$

  which is incorrect, but at least shows some understanding of how to use this formula. No points were awarded for missuses of this formula such as:

$$R = \frac{\rho l}{A} = \frac{\rho(R_2 - R_1)}{4\pi(R_2^2 - R_1^2)},$$

  which is clearly wrong because as $R_1 \to R_2$, the resistance blows up, which doesn't make any sense.

- +2 or +3 for trying to set up an integral for adding the resistors in series and making some mistakes.