Q1 – True/False:

TFTFF; FFTTT; TTFTT; TTTTT

Q2. $A^TA = \begin{bmatrix} 6 & 3 \\ 3 & 9 \end{bmatrix}$, $(A^TA)^{-1} = \frac{1}{15}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

$$A(A^TA)^{-1}A^T = \frac{1}{15}\begin{bmatrix} 7 & 6 & 2 & 4 \\ 6 & 8 & -4 & 2 \\ 2 & -4 & 12 & 4 \\ 4 & 2 & 4 & 3 \end{bmatrix}$$ is the ortho. projection onto $Col(A)$

$\vec{p} = \frac{1}{15}\begin{bmatrix} 19 \\ 12 \\ 14 \\ 13 \end{bmatrix}$, $\vec{v} - \vec{p} = \frac{1}{15}\begin{bmatrix} -4 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, check that

$A^T(\vec{v}-\vec{p}) = \vec{0}$. This must be the case because $(\vec{v}-\vec{p})$ is $\perp$ $Col(A)$, by definition of the orthogonal projection $\vec{p}$ of $\vec{v}$ onto $Col(A)$:
- $\vec{p}$ is in $Col(A)$
- $(\vec{p}-\vec{v}) \perp Col(A)$

Q3. Diagonalize the matrix – let's call it M – and solve by the eigenvector method.

Charpoly: $(\lambda-1)(\lambda^2-\lambda-6)$ (expansion along bottom row)

Eigenvals: $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$

Eigenvects: $\vec{v}_1 = \begin{bmatrix} -2 \\ -2 \\ \ldots \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ \ldots \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ \ldots \end{bmatrix}$.

Now, $\vec{x}(0) = \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-2\\1\end{bmatrix} + \begin{bmatrix}3\\3\\0\end{bmatrix} = \vec{v}_1 + 3\vec{v}_3$

so
$$\vec{x}(t) = e^{t}\vec{v}_1 + 3e^{3t}\vec{v}_3 = \begin{bmatrix}-2e^{t}+3e^{3t}\\-2e^{t}+3e^{3t}\\e^{t}\end{bmatrix}$$

$$e^{-t}\vec{x}(t) = \vec{v}_1 + 3e^{2t}\vec{v}_3 \longrightarrow \vec{v}_1 \text{ as } t \to -\infty.$$

Q4. See materials for Test 2.

Q5. The 2π-periodic Fourier series

$$a_0 + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx)$$

has $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$ $(m>0)$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx.$$

Now $f(x)$ is even so the sine integrals vanish; also, $a_0 = 0$ can be seen from cancellation of the areas above/below the x-axis. So we need

$$a_m = \frac{1}{\pi}\int^{\pi} f(x)\cos mx\,dx = \frac{2}{\ }\int^{\pi}\left(\frac{\pi}{2}-x\right)\cos(mx)\,dx$$

(We used $j(x)=j(-x)$ again)

$$= \frac{2}{\pi}\left(\frac{\pi}{2}-x\right)\frac{\sin(mx)}{m}\Bigg|_0^{\pi} + \frac{2}{\pi m}\int_0^{\pi}\sin(mx)\,dx =$$

$$= 0 - \frac{2}{\pi m^2}\cos(mx)\Big|_0^{\pi} = -\frac{2}{\pi m^2}\left((-1)^m-1\right)$$

and that is 0 for even m and $\frac{4}{\pi m^2}$ for odd m, as wanted.

(b) We should look for a 2π-periodic in x Fourier series

$$u(x,t) = u_0(t) + \sum_{m=1}^{\infty}\left(u_m(t)\cos mx + v_m(t)\sin mx\right)$$

(Actually we will be able to find a solution with cosines only, because $j(x)$ is even.)

Substituting in the wave equation after expanding $j(x)$

$$u_0''(t) + \sum_{m=1}^{\infty}\left(u_m''(t)\cos mx + v_m''(t)\sin mx\right) =$$

$$-\sum_{m=1}^{\infty} m^2\left(u_m(t)\cos mx + v_m(t)\sin mx\right)$$

$$+\frac{4}{\pi}\sum_{m\text{ odd}}\frac{t^2}{m^2}\cos mx$$

Matching $\sin(mx)$, $\cos(mx)$ terms on the two sides gives the independent ODEs

$$u_0''(t) = 0, \quad u_m''(t) = -m^2 u_m(t) \quad \text{for } m \text{ even},$$
$$u_m''(t) = -m^2 u_m(t) + \frac{4}{\pi m^2} t^2 \quad \text{for } m \text{ odd},$$

$$v_m''(t) = -m^2 v_m(t).$$

Clearly we can choose $u_m(t) \equiv 0$ for m even, $v_m = 0$ while the inhomogeneous equations are solved by

$$u_m(t) = \frac{4}{\pi m^4} t^2 - \frac{8}{\pi m^6} \quad \text{(undetermined coeffs)}$$

giving a particular solution

$$u(x,t) = \frac{4}{\pi} \sum_{\substack{m \text{ odd} \\ m \geq 1}} \left(\frac{t^2}{m^4} - \frac{2}{m^6} \right) \cos(mx).$$

The general solution would be the sum of the above and any solution of the homogeneous wave equation (2π-periodic in x). //