Mathematics 54.2
Final Exam, 15 May 2013
180 minutes, 120 points

NAME: ______________________ **ID:** ______________________

GSI: ____

INSTRUCTIONS:
You must justify your answers, except when told otherwise.
All the work for a question should be on the respective sheet.

This is a CLOSED BOOK examination, NO NOTES and NO CALCULATORS are allowed.
NO CELL PHONE or EARPHONE use is permitted.
Please turn in your finished examination to your GSI before leaving the room.

Q1	
Q2	
Q3	
Q4	
Q5	
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Question 1. (40 points) Choose the correct answer. You need not justify your answer. Correct answers carry 2 points credit, incorrect answers carry 2 points penalty. You will not get a negative score on any group of five questions.
When not specified, the sizes of matrices and vectors are such that the products and equalities make sense.

T F If A is an $m \times n$ matrix with m pivot columns, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

T F If A is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ has at least two distinct solutions, then for any $\mathbf{c}$, the equation $A\mathbf{x} = \mathbf{c}$ has infinitely many solutions.

T F If A is a 2×2 matrix with zero determinant, then one row of A is a multiple of the other.

T F If $\det A = 0$ then $A^k = 0$ for some k.

T F If A is a 4×4 matrix and B is produced from A by multiplying the columns of A by 4, then $\det B = 4 \cdot \det A$.

T F The non-pivot columns of a matrix are always linearly dependent.

T F If $AB = 0$ for two matrices A, B, then either $A = 0$ or $B = 0$.

T F Right-multiplying a matrix A by a diagonal matrix D with non-zero diagonal entries scales the columns of A.

T F If A, B are square matrices that commute with each other, then $(A + B)(A - B) = A^2 - B^2$.

T F If $AB = I$ and B is square, then A is invertible.

T F If $\dim \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5) = 5$ then the collection $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\}$ cannot be linearly dependent.

T F If an $m \times n$ matrix A is row equivalent to an echelon matrix U with k non-zero rows, then the dimension of the nullspace of A is $n - k$.

T F If a matrix has orthonormal columns, then it also has orthonormal rows.

T F If W is a subspace of $\mathbf{R}^n$, then $\|\mathbf{v} - \mathrm{proj}_W(\mathbf{v})\|^2 = \|\mathbf{v}\|^2 - \|\mathrm{proj}_W(\mathbf{v})\|^2$.

T F If a matrix M has orthonormal columns, then $M^T M = I$.

T F If the $n \times n$ matrix A is not diagonalizable, then there must exist a vector in $\mathbf{R}^n$ which is not a linear combination of real eigenvectors of A.

T F If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

T F If the square matrix A contains two identical columns, then 0 is an eigenvalue of A.

T F If the matrix A is orthogonal, then the linear transformation $\mathbf{x} \mapsto A \cdot \mathbf{x}$ preserves lengths and angles.

T F If A is a real symmetric $n \times n$ matrix, then there exist an orthogonal basis of A-eigenvectors for $\mathbf{R}^n$.

Question 2. (20 points, 12+4+4)
Construct the 4×4 matrix implementing the orthogonal projection in $\mathbf{R}^4$ onto the column space of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Find the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{v} = [1, 1, 1, 1]^T$ onto $\mathrm{Col}(A)$, and check that $A^T(\mathbf{v} - \mathbf{p}) = \mathbf{0}$. Why does that have to be the case (independently of your computation)?

Question 3. (20 points, 16+4)
Find the solution $\mathbf{x}(t)$ for the vector ODE

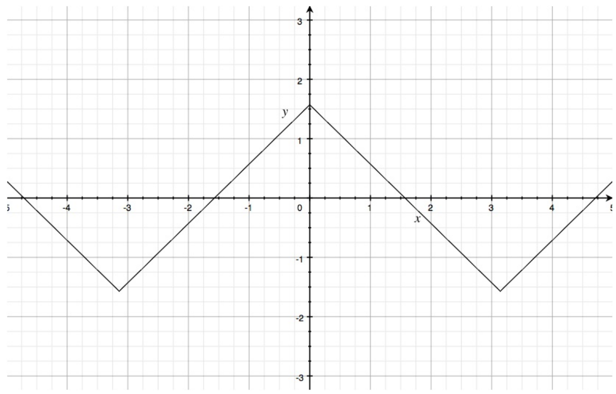
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 4 & 4 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{x}$$

with initial condition $\mathbf{x}(0) = [0, 0, 1]^T$. Explain (and check) your steps.
What is $\lim_{t \to -\infty} e^{-t}\mathbf{x}(t)$?
(*Note:* NOT the limit as $t \to +\infty$.)

Question 4. (15 points)
Let A be a real symmetric matrix, $\lambda \neq \mu$ two distinct (real) eigenvalues, and $\mathbf{v}$ and $\mathbf{w}$ eigenvectors for λ and μ respectively. Show that $\mathbf{v} \perp \mathbf{w}$.

Question 5. (25 points, 10+15)

The 2π-periodic "jagged wave" function defined by $j(x) = \begin{cases} x + \pi/2, & -\pi \le x \le 0 \\ \pi/2 - x, & 0 \le x \le \pi \end{cases}$ is graphed below:



(a) Show (explaining your steps) that $j(x)$ has a 2π-periodic Fourier expansion

$$j(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$

(b) Using Part (a) (or otherwise), write down a *particular solution* of the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + t^2 \cdot j(x)$$

with *periodic boundary conditions*: that is, $u(x + 2\pi, t) = u(x, t)$ for all $x, t \in \mathbf{R}$.
Remarks: You are not required to write out the most general solution, or to match any particular initial conditions at $t = 0$.

THIS PAGE IS FOR ROUGH WORK (not graded unless you mark it otherwise)

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