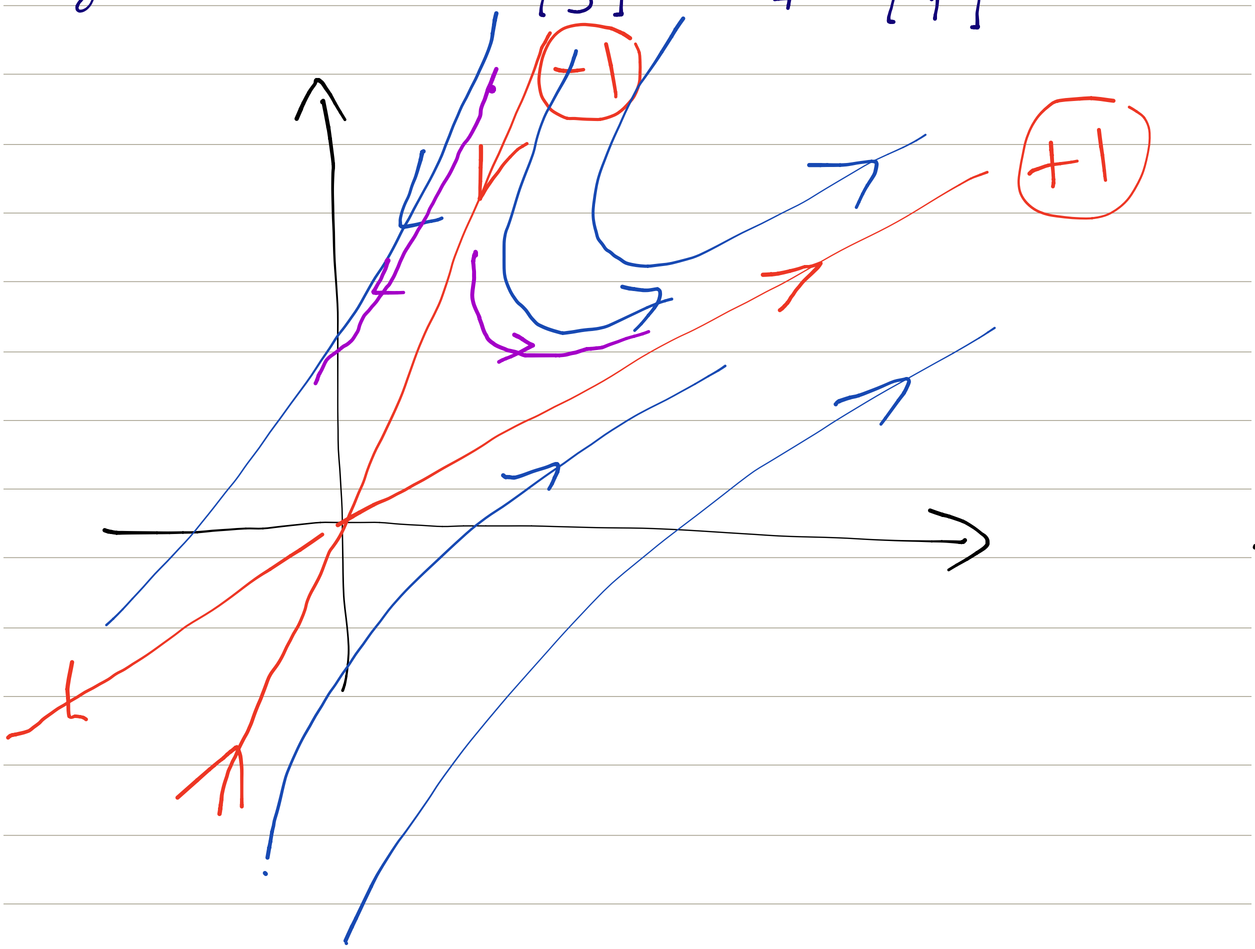
Q1. $d\vec{x}/dt = A\vec{x}$,

$$A = \begin{bmatrix} 1.4 & -0.8 \\ 1.2 & -1.4 \end{bmatrix}$$

Charpoly $\lambda^2 - 1 = 0$, eigenvals ± 1,

eigenvectors $\vec{v}_- = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_+ = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



In the eigenbasis, a vector $\vec{x} = c_+\vec{v}_+ + c_-\vec{v}_-$ evolves as $\vec{x}(t) = c_+ e^t \vec{v}_+ + c_- e^{-t}\vec{v}_-$, so the coordinates $c_+(t) = c_+ e^t$, $c_-(t) = c_- e^{-t}$ satisfy the equations $c_+(t)\cdot c_-(t) = \text{const.}$ of hyperbolae.

(b) The x_1's thrive on their own (if $x_2 = 0$) and suffer from the presence of x_2's.

The x_2's cannot survive on their own (if $x_1 = 0$) but do quite well with x_1's around.

So this looks like a predator - prey relationship with x_1 the prey.

(c) $\begin{bmatrix} 1000 \\ 3500 \end{bmatrix} = 1200\,\vec{v}_- - 100\,\vec{v}_+$

$$\rightsquigarrow 1200e^{-t}\,\vec{v}_- - 100\,e^{t}\,\vec{v}_+ = \begin{bmatrix} 1200\,e^{-t} - 200\,e^{t} \\ 3600\,e^{-t} - 100\,e^{t} \end{bmatrix}$$

so the x_1 die out first, whereupon the x_2 decline as $e^{-1.4t}$.

$\begin{bmatrix} 1000 \\ 2750 \end{bmatrix} = 900\,\vec{v}_- + 50\,\vec{v}_+$

$$\rightsquigarrow 900\,e^{-t}\,\vec{v}_- + 50\,e^{t}\,\vec{v}_+ = \begin{bmatrix} 900\,e^{-t} + 100\,e^{t} \\ 2700\,e^{-t} + 50\,e^{t} \end{bmatrix}$$

always positive and a good approximation to $50e^{t}\,\vec{v}_+$ for large t, so the populations thrive in a 2:1 ratio

Remark: you can also conclude all this from the phase portrait by following the trajectories.

Q2. $\sin x = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x \, dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = \frac{2}{\pi}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] dx$$

$$= \frac{1}{\pi} \frac{-\cos(n+1)x}{n+1}\Big|_0^{\pi} + \frac{1}{\pi} \frac{\cos(n-1)x}{n-1}\Big|_0^{\pi}$$

If n is odd, $n \pm 1$ is even and $\cos(n \pm 1)\pi = 1$ so the integrals vanish.

If n is even, then $\cos(n \pm 1)\pi = -1$ and we get

$$\frac{1}{\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right] = \frac{4}{\pi(1-n^2)}$$

$$\sin x = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4 \cos(2kx)}{\pi(1-n^2)}$$

Q3. First,

$$\sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x = \frac{1}{2} \sin 2x \cos 2x \cos 4x$$

$$= \frac{1}{4} \sin 4x \cos 4x = \frac{1}{8} \sin 8x.$$

So the initial condition is $\sin x + \frac{1}{8}\sin 8x$.
Next, the boundary conditions require a sine series expansion of $u(x,t)$,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(nx)$$

with general solution $u_n(t) = u_n(0)\, e^{-n^2 t}$.
At $t=0$, we need

$$\sum_{n=1}^{\infty} u_n(0)\sin nx = \sin x + \frac{1}{8}\sin 8x$$

so $u_1(0) = 1$, $u_8(0) = \frac{1}{8}$ and the others vanish and so

$$u(x,t) = e^{-t}\sin(x) + \frac{1}{8}e^{-64t}\sin(8x)$$

Q4. (More involved than required for full credit)
Lagrange's method for solving

$$y''(t) + b\cdot y'(t) + c\cdot y(t) = f(t)$$

(can even be used when b, c depend on t) goes as follows. Let $y_1(t), y_2(t)$ be two fundamental sols. of the homogeneous equation

$$y''(t) + b\cdot y'(t) + c\cdot y(t) = 0$$

and write, for a particular solution $y_p(t)$,

$$\begin{bmatrix} y_p(t) \\ y_p'(t) \end{bmatrix} = v_1(t) \begin{bmatrix} y_1(t) \\ y_1'(t) \end{bmatrix} + v_2(t) \begin{bmatrix} y_2(t) \\ y_2'(t) \end{bmatrix}$$

with unknown v_1, v_2.
The second equation $y_p' = v_1 \cdot y_1' + v_2 y_2'$, when compared with the Leibniz rule applied to $y_p = v_1 y_1 + v_2 y_2$, gives

$$y_1 v_1' + y_2 v_2' = 0. \qquad (A)$$

On the other hand, substituting $y_p'' = (v_1 y_1' + v_2 y_2')'$ into the original ODE gives

$$y_1' v_1' + y_2' v_2' = f \qquad (B)$$

From (A) + (B) we can solve for v_1', v_2' as

$$v_1' = -f y_2 / (y_1 y_2' - y_1' y_2)$$
$$v_2' = f y_1 / (y_1 y_2' - y_1' y_2).$$

In our case we can take $y_1 = e^t, y_2 = e^{-t}$ and get

$$v_1' = 1/(1+e^t), \quad v_2' = -e^{2t}/(1+e^t)$$

With the substitution $u = e^t$ we get

$$v_1 = \int \frac{dt}{1+e^t} = \int \frac{du}{u(1+u)} = \int \frac{du}{u} - \int \frac{du}{1+u} = \ln u - \ln(1+u)$$

$= t - \ln(1+e^t);$

$$v_2 = -\int \frac{e^{2t}dt}{1+e^t} = -\int \frac{u^2 du}{u(1+u)} = -\int \frac{u\,du}{1+u} = -u + \ln(1+u) =$$

$$= \ln(1+e^t) - e^t$$

Therefore $y_p = v_1 e^t + v_2 e^{-t} = te^t - 1 - (e^t - e^{-t})\ln(1+e^t)$

and the general solution is $y_p + Ae^t + Be^{-t}$.