UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME132 Dynamic Systems and Feedback

Midterm I — Spring 2013

Closed Book and Closed Notes. One 8.5 × 11 sheet (double-sided) of handwritten notes allowed. Scientific calculator with no graphics allowed.

Your Name:

Please answer all questions. This exam has 10 pages.

Problem:	1	2	3	4	5	Total
Max. Grade:	20	10	15	40	15	100
Grade:	24	9	15	40	15	103

P1. Consider the following input-output differential equation:

24/20

$$\ddot{y}(t) + e^{2t}\dot{y}(t) - y(t) = u(t)$$

where $u(t)$ is the input and $y(t)$ is the output. The initial conditions are $\dot{y}(0) = 0$ and $y(0) = 1$.

(a) Please answer the following questions and give brief explanations (or derivations) to justify your answers.

i. Is the system linear?

Yes. All terms of y or $\dot{y}$ or $\ddot{y}$ are connected by linear operators (no cross-terms such as $y \cdot \dot{y}$)

(+1)

ii. Is it time-invariant?

No. coefficient of $\dot{y}$ is dependent on t.

(+1)

iii. Is it dynamic?

Yes. contains $\dot{y}$ and $\ddot{y}$ therefore depends on previous states.

(+1)

(b) **Write down the differential equation in state-space form. Specify also the initial conditions of the state-space model. (Hint: your state-space matrices might include non-constant terms)**

$$\ddot{y} + e^{2t}\dot{y} - y = u \qquad \dot{y}(0) = 0, \quad y(0) = 1$$

Let $\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases}$ $\quad \dot{x}_1 = \dot{y} = x_2$

$\dot{x}_2 = \ddot{y} = -e^{2t}\dot{y} + y + u$

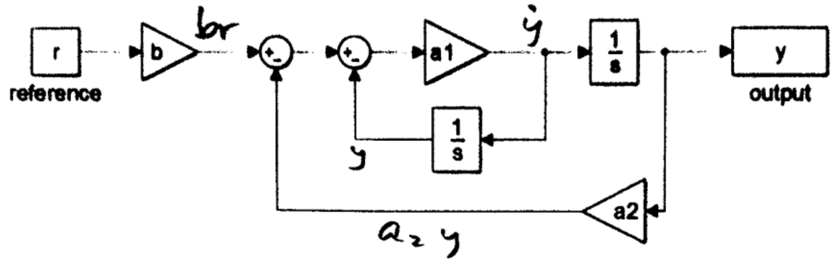
$\dot{x}_2 = -e^{2t}x_2 + x_1 + u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -e^{2t} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,u$$

IC: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (+1)

P2. Consider the Simulink block diagram below, with $b \neq 0$, $a_1 \neq 0$ and $a_2 \neq 0$.



Assume the initial conditions of both the integrator blocks ($\frac{1}{s}$) are zero. Write the SLODE model (linking y to r) for this diagram.

$$(br - a_2 y - y)\,a_1 = \dot{y}$$

$$a_1 b r - a_1 a_2 y - a_1 y = \dot{y}$$

$$\boxed{\dot{y} = -(a_1 + a_1 a_2)\,y + a_1 b r}$$

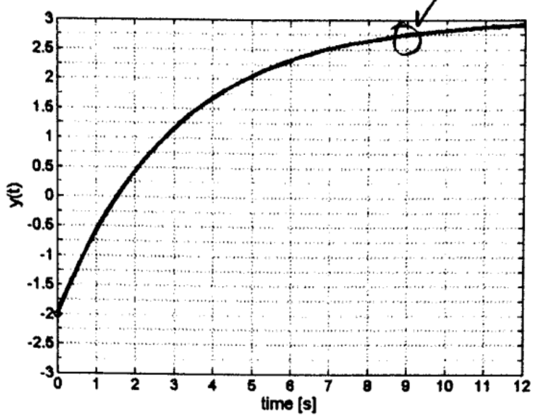
I.C. $\begin{cases} y(0) = 0 \\ \cancel{\dot{y}(0) = 0} \end{cases}$

-1

P3. The plot below depicts the response $y(t)$ of a first order SLODE

$$\dot{y} + ay = bu, \quad y(0) = -2$$

when the input is a unit step ($u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t \geq 0$).



15/15

Determine a and b. (Note 1: the output steady-state value is 3. Note 2: **try to be as accurate as you can.**)

$$y_{ss} = 3 = \frac{b}{a} u_{ss} = \frac{b}{a} \cdot 1 \Rightarrow \frac{b}{a} = 3$$

$$y_{tr} = e^{-at}(y_0 - y_{ss}) + y_0$$

at $t = 9$: $\quad 2.75 = e^{-a \cdot 9}(-2 - 3) + 3$

$$\boxed{a = 0.333 \approx \tfrac{1}{3}}$$

$$\boxed{b = 3a = 1}$$

P4. A (static) system is modeled by the equation:

$$y = Gu$$

where u is the input, y is the output and G is a constant. We wish to control the system by using a controller of the form

$$\dot{u} = K_1 u + K_2(r - y)$$

10/10

where $r(t)$ is the reference signal, and K_1 and K_2 are constants.

(a) Write the closed-loop SLODE model.

$\frac{d}{dt}(y = Gu)$: $\dot{y} = G\dot{u} = G(K_1 u + K_2(r-y))$

$u = \frac{y}{G}$

$\dot{y} = GK_1 u + GK_2 r - GK_2 y$

$\dot{y} = \cancel{G}K_1 \frac{y}{\cancel{G}} + GK_2 r - GK_2 y$

$$\boxed{\dot{y} + (GK_2 - K_1) y = GK_2 r}$$

(b) Assume $K_2 = K_1$ and a perfect model knowledge with $G = 20$. Design a controller K_1 such that

(1) the closed-loop system is stable,

(2) the steady-state output tracking error is less than 10% when the reference signal is a constant ($r(t) = \bar{r}$ for $t \geq 0$), i.e. $|y_{ss} - \bar{r}| < 0.1\,|\bar{r}|$ where y_{ss} is the steady-state output.

(3) The closed-loop settling-time is smaller than 0.3 seconds. (Use settling time equal to three times the closed-loop time-constant.)

$$\dot{y} + (GK_2 - K_1)\, y = GK_2\, r$$

10/10

for $K_2 = K_1$ and $G = 20$:

$$\dot{y} + 19K_1\, y = 20K_1 \cdot r$$

(1) Stable : $19K_1 > 0 \Rightarrow K_1 > 0$

(2) tracking error : $y_{ss} = \frac{b}{a} u_{ss} = \frac{20K_1\bar{r}}{19K_1} = \frac{20}{19}\bar{r}$

good

$$|y_{ss} - \bar{r}| < 0.1\,|\bar{r}|$$

$$\left|\frac{20}{19}\bar{r} - \bar{r}\right| < 0.1\,|\bar{r}|$$

$\frac{1}{19} < 0.1$ is true regardless of K_1

$\therefore$ (2) will be true for all K_1

(3) $3T = \frac{3}{a} < 0.3$

$$\frac{3}{19K_1} < 0.3$$

$$K_1 > 0.526$$

$\checkmark$

$\therefore$ a possible value for K_1 is $K_1 = 1 = K_2$

(c) Use K_1 (and K_2) you found in the previous question. What is the effect of a 50% decrease in the model G on the tracking error?

let $K_1 = K_2 = 1$

10/10

$$\dot{y} + (G-1)\,y = G\,r$$

$$\Delta = \text{tracking error} = \left| \frac{G\bar{r}}{G-1} - \bar{r} \right|$$

$$= \frac{1}{G}\,\bar{r}$$

$$\Delta \propto \frac{1}{G}$$

✓

$\therefore$ a 50% decrease in G would increase Δ by factor of 2 (ie $\Delta(0.5G) = 2 \times \Delta(G)$)

for $K_1 = K_2 = 1$

NOT PRECISE

$G-1$

(d) Remove the assumption that $K_2 = K_1$. Can you design K_1 and K_2 such that the steady-state tracking error is 0 for any value of G?

$$\dot{y} + (GK_2 - K_1)y = GK_2 r$$

10/10

$$\text{error} = \frac{GK_2\bar{r}}{GK_2 - K_1} - \bar{r} = 0$$

$$\frac{GK_2}{GK_2 - K_1} = 1$$

$$GK_2 = GK_2 - K_1$$

$$K_1 = 0$$

Stability requires $GK_2 > 0$

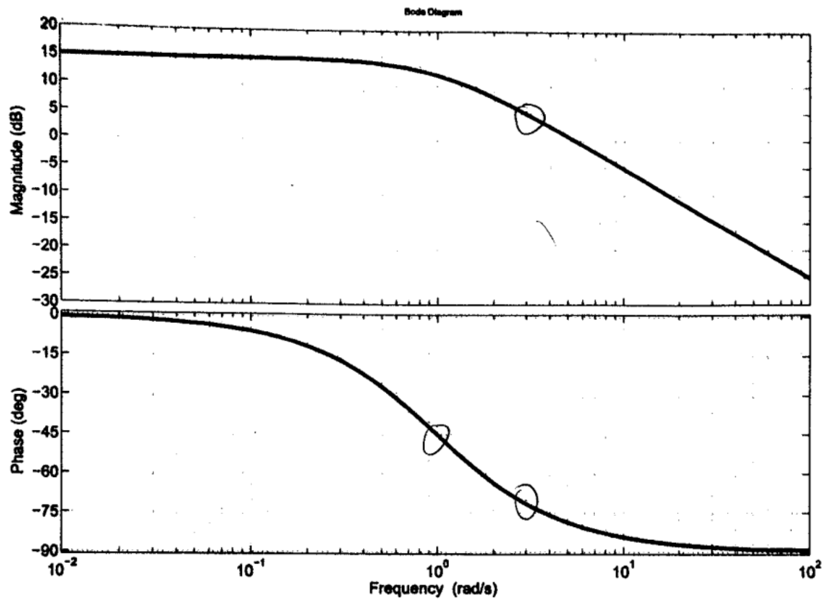
as long as K_2 has the same sign as G ($K_2, G \neq 0$) (both positive or both negative) and $K_1 = 0$, error will be zero

good

P5. Consider the first order SLODE

$$\dot{y}(t) + ay(t) = bu(t)$$

where $a > 0$ and $b > 0$. The bode plot for this system is shown below.



Write the steady state output $y_{ss}(t)$ of the SLODE when the input u(t) is

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 + 5\sin(3t) & t \geq 0 \end{cases}.$$

(ps. just to be clear, I am not asking you to sketch it, just report the analytical solution)

(let $u_1 = 1$, $u_2 = 5\sin(3t)$)

solutions are superimposable $y_{ss} = y_{ss_1} + y_{ss_2}$

from the ϕ vs ω plot. $\Phi = -45°$ when $\omega = 1$

$$\tan^{-1}\left(\frac{-1}{a}\right) = -45$$

$$a = 1$$

from the MdB vs ω plot $M\,dB = 5$ when $\omega = 3$

$$M = 1.7783 = \frac{b}{\sqrt{a^2+\omega^2}} = \frac{b}{\sqrt{1^2+3^2}} \Rightarrow b = 5.623$$

$$y_{ss_1} = \frac{b}{a} u_{ss} = \frac{5.623}{1} \times 1 = 5.623$$

next page

$y_{ss2} = AM\sin(\omega t+\phi)$

$\omega = 3$. $A = 5$. $M = 1.7783$ as found previously for $\omega = 3$

from ϕ vs ω plot at $\omega = 3$, $\phi \approx -70^\circ = -1.222$ rad

$$y_{ss2} = 5\times 1.7783 \sin(3t - 1.222)$$

$$\boxed{y_{ss} = 5.623 + 8.8915\sin(3t - 1.222)}$$