NAME: 3 April 2009

Physics 112

Spring 2009

Midterm 2

(50 minutes =50 points)

Time yourself and move to the next problem after a number of minutes equal to the number of points.

1. Entropy (10 points):

a) (5 points) Show from our general definition of the entropy (which applies both to isolated and non-isolated systems) that the entropy of a system in contact with a thermal bath at temperature τ, is

$$\sigma = \frac{\partial(\tau \log Z)}{\partial \tau}$$

Starting from $\sigma = -\sum_{\text{states } s} p_s \log p_s$ $\quad p_s = \frac{1}{Z} e^{-\frac{\varepsilon_s}{\tau}}$ $\quad \sum_s p_s = 1$

$Z = \sum_s e^{-\frac{\varepsilon_s}{\tau}}$ $\quad \sum_s p_s \varepsilon_s = \sum_s \frac{1}{Z} e^{-\frac{\varepsilon_s}{\tau}} \varepsilon_s = +\tau^2 \frac{\partial \log Z}{\partial \tau}$

$$\sigma = \sum_s p_s \log Z + \sum_s p_s \frac{\varepsilon_s}{\tau} = \log Z + \tau \frac{\partial \log Z}{\partial \tau} = \frac{\partial(\tau \log Z)}{\partial \tau}$$

b) (5 points) Why is this not the logarithm of the total number of states?

The system is not isolated, and therefore the states are not equiprobable.

Only when $p_s = \frac{1}{g_t}$ $\forall s$ (g_t = total number of states)

$$\sigma = \log g_t$$

2. Paramagnetism and adiabatic demagnetization (20 points)

Let us consider a system of N_S distinguishable spin 1 sites in a magnetic field B at temperature τ. Each spin has magnetic moment m and its energy in the magnetic field is $\varepsilon_+ = -mB$, $\varepsilon_o = 0$ and $\varepsilon_- = mB$, depending whether it points along, perpendicular to or opposite to the magnetic field.

a) (5 points) Write down the partition function of each spin and probabilities of it pointing along, perpendicular to or opposite to the magnetic field.

$$Z_1 = \underset{\varepsilon_o}{1} + \underset{\varepsilon_+}{e^{-\frac{(-mB)}{\tau}}} + \underset{\varepsilon_-}{e^{-\frac{(mB)}{\tau}}} = 1 + e^{\frac{mB}{\tau}} + e^{-\frac{mB}{\tau}} = 1 + 2\cosh\frac{mB}{\tau}$$

$$\text{prob}(\varepsilon_o) = \frac{1}{1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}} = \frac{1}{1+2\cosh\frac{mB}{\tau}}$$

$$\text{prob}(\varepsilon_+) = \frac{e^{\frac{mB}{\tau}}}{1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}} = \frac{e^{\frac{mB}{\tau}}}{1+2\cosh\frac{mB}{\tau}}$$

$$\text{prob}(\varepsilon_-) = \frac{e^{-\frac{mB}{\tau}}}{1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}} = \frac{e^{-\frac{mB}{\tau}}}{1+2\cosh\frac{mB}{\tau}}$$

b) (10 points) From the partition fnnction of a single spin, deduce that the total entropy of the system is (at small mB/τ)

$$\sigma_s = N_s(\log 3 - \frac{m^2B^2}{3\tau^2}).$$

You may want to remember that for small ε,

$$\log(1+\varepsilon) \sim \varepsilon + O(\varepsilon^2), \qquad \cosh(\varepsilon) \sim 1 + \frac{\varepsilon^2}{2} + O(\varepsilon^3).$$

$$\frac{1}{1+\varepsilon} \sim 1 - \varepsilon + O(\varepsilon^2), \qquad \sinh(\varepsilon) \sim \varepsilon + O(\varepsilon^2)$$

$$\exp(\varepsilon) = 1 + \varepsilon + \frac{\varepsilon^2}{2} + O(\varepsilon^3).$$

From problem 1, for 1 spin

$$\sigma_1 = \frac{\partial(\tau \log Z_1)}{\partial \tau} = \log Z_1 + \tau\frac{\partial \log Z_1}{\partial \tau}$$
$$= \log\left(1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}\right) + \frac{-\frac{mB}{\tau}e^{\frac{mB}{\tau}} + \frac{mB}{\tau}e^{-\frac{mB}{\tau}}}{1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}}$$

$$e^{\frac{mB}{\tau}} = 1 + \frac{mB}{\tau} + \frac{1}{2}\left(\frac{mB}{\tau}\right)^2 + O\left[\left(\frac{mB}{\tau}\right)^3\right]$$

$$e^{-\frac{mB}{\tau}} = 1 - \frac{mB}{\tau} + \frac{1}{2}\left(\frac{mB}{\tau}\right)^2 + O\left[\left(\frac{mB}{\tau}\right)^3\right]$$

$$1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}} = 3 + \left(\frac{mB}{\tau}\right)^2 = 3\left[1+\frac{1}{3}\left(\frac{mB}{\tau}\right)^2\right] + O\left[\left(\frac{mB}{\tau}\right)^4\right]$$

$$\log\left(1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}\right) = \log 3 + \frac{1}{3}\left(\frac{mB}{\tau}\right)^2 + O\left[\left(\frac{mB}{\tau}\right)^4\right]$$

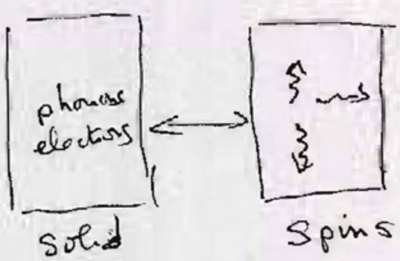
$$-\frac{mB}{\tau}\,\frac{\left(e^{\frac{mB}{\tau}}-e^{-\frac{mB}{\tau}}\right)}{1+e^{\frac{mB}{\tau}}+e^{-\frac{mB}{\tau}}} = -\frac{mB}{\tau}\,\frac{\left(\frac{2mB}{\tau}\right)}{3} + O\left[\left(\frac{mB}{\tau}\right)^4\right]$$

finally
$$\sigma_1 = \log 3 + \left(\frac{mB}{\tau}\right)^2\left(\frac{1}{3}-\frac{2}{3}\right) + O\left[\left(\frac{mB}{\tau}\right)^4\right]$$

$$= \log 3 - \frac{1}{3}\left(\frac{mB}{\tau}\right)^2 + O\left[\left(\frac{mB}{\tau}\right)^4\right]$$

For N_S distinguishable spin
$$\sigma_{N_s} = N_S\left[\log 3 - \frac{1}{3}\left(\frac{mB}{\tau}\right)^2\right]$$

c) (5 points) The spin system is part of a larger system (e.g., the solid containing the spins) that we suppose isolated from the outside. We decrease reversibly the magnetic field to zero. Use the conservation of entropy to show that the solid temperature decreases. This method is used in adiabatic demagnetization refrigerators, which use salt pills in superconducting solenoids and typically reach 100mK temperatures.



When we decrease adiabatically the magnetic field B, the entropy of the spins tends to increase - Therefore the entropy of the solid has to decrease. The temperature of the solid and spins decrease !

For example: if the heat capacity of the solid is

$$C = C_0 T^3 \quad \text{(insulator, dominated by phonons)}$$

$$dU = C\,dT \qquad dS = \frac{1}{T}\,dU = c_0 T^2\,dT$$

$$S_{solid} = \frac{C_0 T^3}{3} \qquad \sigma_{sol} = \frac{C_0\,\tau^3}{3k_b^2}$$

$$\frac{C_0 T_i^3}{3} + k_b N_S\left(\log 3 - \frac{1}{3}\left(\frac{mB}{k_b T_i}\right)^2\right) = \frac{C_0 T_f^3}{3} + k_B N_S \log 3$$

$$T_f = \left[T_i^3 - \frac{3k_b N_S}{C_0}\left(\frac{mB}{k_B T_i}\right)^2\right]^{1/3}$$

3. Fermi-Dirac Particles (20 points)

We will see later in the course that two half-integer spin particles cannot be in the same quantum state (Pauli exclusion principle). Therefore in one state s of energy ε_s per particle with one particular spin orientation, we can have either zero or one particle. Let consider such a state in equilibrium with a much larger system at temperature τ, with which it can exchange energy and particles.

a) (3 points) Write down the grand partition function of this state as a function of ε_s, τ and the chemical potential μ.

$$\mathcal{Z} = \underset{\substack{\uparrow \\ \text{0 particle}}}{1} + \underset{\substack{\uparrow \\ \text{1 particle}}}{e^{-\frac{\varepsilon-\mu}{\tau}}} \quad \text{in the state}$$

b) (3 points) What is the general expression of the mean number of particles in a state in term of a partial derivative of the grand partition function? Justify your answer (e.g. does not just copy it from your notes)

Starting from $\mathcal{Z} = \sum_{s,N_s} e^{-\frac{(\varepsilon_s-\mu)N_s}{\tau}}$

we see that $\tau \frac{\partial \log \mathcal{Z}}{\partial \mu} = \frac{1}{\mathcal{Z}} \sum_{s,N_s} e^{-\frac{(\varepsilon_s-\mu)N_s}{\tau}} N_s$

$$\Rightarrow \quad \langle N_s \rangle = \tau \frac{\partial \log \mathcal{Z}}{\partial \mu}$$

c) (4 points) Deduce from this result that the mean number of particles in state s with energy ε_s per particle (and one spin orientation) is for half-integer spin particles

$$\langle N_s \rangle = \frac{1}{\exp\left(\frac{\varepsilon_s - \mu}{\tau}\right) + 1}$$

$$\langle N_s \rangle = \tau \frac{\partial \log \mathcal{Z}}{\partial \mu} = \frac{e^{-\frac{(\varepsilon-\mu)}{\tau}}}{1 + e^{-\left(\frac{\varepsilon-\mu}{\tau}\right)}}$$

or $\langle N_s \rangle = \dfrac{1}{e^{\frac{\varepsilon-\mu}{\tau}} + 1}$

d) (5 points) In order to know the total number of particles in the system we have to sum over the density of states

$$\langle N\rangle = \sum_{s,spins} \frac{1}{\exp\left(\frac{\varepsilon_s-\mu}{\tau}\right)+1}$$

We will call g_s is the number of independent spin states (2 for spin 1/2), and perform the appropriate integral over phase space to sum over spatial quantum states. Express <N> as a function of g_s, the volume V and an integral over d^3p.

The density of spatial states is $\frac{d^3x\, d^3p}{h^3}$

$$< N > = \sum_{s,\,spins} \frac{1}{\exp\left(\frac{\varepsilon_s-\mu}{\tau}\right)+1} = g_s \int \frac{d^3x\, d^3p}{h^3} \frac{1}{\exp\left(\frac{\varepsilon(p)-\mu}{\tau}\right)+1}$$

$$= \frac{g_s V}{h^3} \int \frac{d^3p}{\exp\left(\frac{\varepsilon(p)-\mu}{\tau}\right)+1}$$

e) (5 points) What is the condition for $\langle N_s\rangle \ll 1$? If this low occupation number is correct for most of the states show that for non relativistic particles

$$\langle N\rangle = g_s V \exp\left(\frac{\mu}{\tau}\right)\left(\frac{2\pi M\tau}{h^2}\right)^{\frac{3}{2}} = g_s V \exp\left(\frac{\mu}{\tau}\right) n_Q$$

$$\text{or } \mu = \tau \log\left(\frac{n}{g_s n_Q}\right) \text{ where } n = \frac{\langle N\rangle}{V}$$

$$\varepsilon_s = \frac{p^2}{2m} = \frac{p_x^2+p_y^2+p_z^2}{2m}$$

This is the classical result (the g_s factor comes from the spin degrees of freedom and is also present classically). This rigorous result justifies the Gibbs *ansatz* of dividing by *N!* the naïve partition function for a system of *N* undistinguishable particles.
In this derivation, you may want to use the fact that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right)dx = \sqrt{2\pi}\sigma$$

If $N_s \ll 1$ $\quad \frac{1}{\exp\left(\frac{\varepsilon(p)-\mu}{\tau}\right)+1} \sim \exp\left(-\frac{\varepsilon(p)-\mu}{\tau}\right)$

$$\Rightarrow <N> = \frac{g_s V}{h^3} e^{\frac{\mu}{\tau}} \int_0^\infty p^2\, dp\, d\Omega \, \exp\left(-\frac{p^2}{2m\tau}\right)$$

that we can integrate by part $\quad \frac{g_s V}{h^3} e^{\frac{\mu}{\tau}} 4\pi \left(\underbrace{\cancel{\left[-m\tau\, p \exp\left(-\frac{p^2}{2m\tau}\right)\right]_0^{+\infty}}}_{=0} + m\tau \underbrace{\int_0^{+\infty} dp\, \exp\left(-\frac{p^2}{2m\tau}\right)}_{\frac{1}{2}\sqrt{2\pi m\tau}} \right)$

$$\Rightarrow <N> = g_s V \exp\left(\frac{\mu}{\tau}\right)\left(\frac{2\pi m\tau}{h^2}\right)^{3/2}$$

or more simply

$$<N> = \frac{g_s V}{h^3} e^{\frac{\mu}{\tau}} \underbrace{\int_{-\infty}^{+\infty} dp_x\, e^{-\frac{p_x^2}{2m\tau}}}_{=\sqrt{2\pi m\tau}} \underbrace{\int_{-\infty}^{+\infty} dp_y\, e^{-\frac{p_y^2}{2m\tau}}}_{=\sqrt{2\pi m\tau}} \underbrace{\int_{-\infty}^{+\infty} dp_z\, e^{-\frac{p_z^2}{2m\tau}}}_{=\sqrt{2\pi m\tau}}$$

$$<N> = g_s V \exp\left(\frac{\mu}{\tau}\right)\left(\frac{2\pi m\tau}{h^2}\right)^{3/2} = g_s V \exp\left(\frac{\mu}{\tau}\right) n_Q$$