Mathematics 54.1
Midterm 1, 1 October 2009
80 minutes, 80 points

NAME: ______________________________ **ID:** ______________________________

GSI: ____

INSTRUCTIONS:
You must justify your answers, except when told otherwise.
All the work for a question must be on the respective sheet.
This is a CLOSED BOOK examination and NO CALCULATORS are allowed.

Question 1. (18 points)
True or False? Circle the correct answer, no justification necessary. Correct answers carry 1.5 points, wrong answers carry 1.5 points penalty. However, you will not get a negative total score on any group of six questions.

T F If A is an $m \times n$ matrix and the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbf{R}^m$, then A has m pivots.

T F The second row of AB is the seond row of A multiplied on the right by B.

T F If the matrix A is invertible, then A^{-1} is also invertible, and its inverse is A itself.

T F The columns of any 4×5 matrix are linearly dependent.

T F The columns of an invertible $n \times n$ matrix form a basis of $\mathbf{R}^n$.

T F The determinant of any square matrix A is the product of its diagonal entries.

T F If a finite set S of vectors spans a vector space V, then some subset of S is a basis of V.

T F The dimensions of the row space and of the column space of a matrix A are the same, whether or not A is square.

T F If the equation $A\mathbf{x} = \mathbf{0}$ admits a non-trivial solution, then the columns of the matrix A are linearly dependent.

T F The dimension of the null space of a matrix is equal to the number of columns that do *not* contain pivots.

T F A linear transformation from $\mathbf{R}^5$ to $\mathbf{R}^6$ cannot be onto.

T F If none of the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbf{R}^3$ is a multiple of one of the other vectors, then S is linearly independent.

Question 2. (14 pts, 4+6+4)
(a) Define what is meant by a linear transformation $T : V \to W$ between vector spaces V, W.
(b) Define the *kernel* $\ker(T)$ (also called *null-space*) and the *range* $\mathrm{Ran}(T)$ of a linear transformation $T : V \to W$, and show that they are linear subspaces of V and W, respectively. In the process, you should explain the meaning of 'linear subspace'.
(c) Assume that V is 2-dimensional and let $\mathbf{v}_1, \mathbf{v}_2$ be a basis of V. If $\ker(T) = 0$, show that $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are linearly independent vectors in W.

Question 3. (12 pts)
For the following matrix, find bases for the row space, column space, nullspace and left nullspace. Make your procedure clear.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 4 & 5 \end{bmatrix}$$

Question 4. (14 pts, 3+4+5+2)
Let P_3 be the space of polynomials of degree no more than 3. Define the mapping $T : P_3 \to P_3$ which sends the polynomial $p(x) \in P_3$ to the polynomial

$$Tp(x) = x^2 \cdot p(1) - 2p'(x) - p''(x).$$

(For instance, the polynomial x^2 is sent to $x^2 - 4x - 2$.)
Show that T is a linear map. Find the matrix representing T in the basis $\{1, x, x^2, x^3\}$ of P_3. Find bases for the kernel and range of T. Does the polynomial $x^2 + x + 1$ belong to the range of T?

Question 5. (12 pts)
For each of the following matrices, find the inverse and verify it, or explain why it is not invertible:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$$

Question 6. (10 pts)

(a) Show that if the square matrix A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

(b) If $\det(A^5) = 0$, explain why A cannot be invertible.

THIS PAGE IS FOR ROUGH WORK (not graded)