1

The electric field in a capacitor is

$$E = \frac{\sigma}{\varepsilon_0}$$

Using the fact that for parallel plates, $Q = CV = \frac{A}{\varepsilon_0 d}V$ we have $E = \frac{V}{\varepsilon_0^2 d}$.
Now, summing the forces in the x and y directions we have the equations:

$$\begin{aligned} mg &= T\cos\theta \\ qE &= T\sin\theta \end{aligned}$$

It follows that:

$$\frac{qE}{mg} = \tan\theta = \frac{qV}{mg\varepsilon_0^2 d}$$

This specifies the magnitude of q. We know from the direction of the battery that q must be negative.

2

3

The expression for the potential can be written down immediately:

$$V(x) = \int \frac{kdq}{r} = \int_0^{2\pi}\int_0^{\theta_0} \frac{k\sigma R^2 \sin\phi}{\sqrt{R^2 + x^2 - 2Rx\cos(\pi-\phi)}} d\phi d\theta = 2\pi k\sigma \int_0^{\theta_0} \frac{\sin\phi}{\sqrt{1 + \left(\frac{x}{R}\right)^2 - 2\frac{x}{R}\frac{\cos(\pi-\phi)}{R}}} d\phi$$

We can actually do this integral exactly but it is easier to first use the expansion given in the problem on the denominator.

$$V(x) \approx 2\pi k\sigma \int_0^{\theta_0} \sin\phi\left(1 + \frac{x}{R}\frac{\cos(\pi-\phi)}{R}\right)d\phi = 2\pi k\sigma\left(1 - \cos\theta_0 - \frac{x}{2R^2}\sin^2\theta_0\right)$$

$$= 2\pi k\sigma(1-\cos\theta_0)\left[1 - \frac{x\sin^2\theta_0}{2R^2(1-\cos\theta_0)}\right] = 2\pi k\sigma(1-\cos\theta_0)\left[1 - \frac{(1+\cos\theta_0)}{2R}\frac{x}{R}\right]$$

Thus we have that

$$V_0 = 2\pi k\sigma(1-\cos\theta_0) \qquad \text{and} \qquad \alpha = -\frac{(1+\cos\theta_0)}{2R}$$

To find the electric field we first notice that it must be in the x-direction by symmetry. Thus we only need the gradient in the $\hat{x}$ direction and it suffices to take the derivative with respect to x, and then set $x = 0$. This gives:

$$\vec{E} = \frac{V_0\alpha}{R}\hat{x}$$

4

To do this problem we will use superposition of electric fields and assume that ρ is positive (WLOG). The electric field at some radial distance $r < R$ from the cylinder with no hole is given by Gauss' Law. Take a gaussian cylinder of length l parallel to the axis. Then:

$$\oint \vec{E}\cdot d\vec{a} = 2\pi r l E = \frac{Q}{\varepsilon_0} = \frac{\pi r^2 l\rho}{\varepsilon_0} \implies \vec{E} = \frac{r\rho}{2\varepsilon_0}\hat{r}$$

Likewise if we position the field of the missing sphere of charge at the origin it is:

$$\vec{E} = \frac{4/3\pi \left(\frac{R}{2}\right)^3 (-\rho)}{\varepsilon_0 4\pi r^2}\hat{r} = -\frac{\left(\frac{R}{2}\right)^3 \rho}{3\varepsilon_0 r^2}\hat{r} \qquad (r > R/2)$$

Where the electric field is due to an oppositely charged sphere of the same charge density. Also note that the field at the center of a sphere is 0.
Now we are ready to add the fields. At point A the field of the cylinder is 0 because $r = 0$, but the field of the sphere points in the positive direction:

$$\vec{E_A} = \frac{\left(\frac{R}{2}\right)\rho}{3\varepsilon_0}\hat{x}$$

At position B, the cylinder field is pointing in the positive x direction and the sphere field is pointing in the negative y direction:

$$\vec{E_B} = \frac{\left(\frac{R}{2}\right)\rho}{3\varepsilon_0}(-\hat{y}) + \frac{R\rho}{4\varepsilon_0}\hat{x}$$

Atpoint C, the sphere field is 0 and the cylinder field is the same as B:

$$\vec{E_C} = \frac{R\rho}{4\varepsilon_0}\hat{x}$$

PHYSICS 7B FALL 2012 SEC 2/3 PROBLEM 4 RUBRIC

This problem has three main parts: finding the electric field due to the two simple configurations, and then superposing these fields at the three points A, B, and C. Points are awarded as follows:

Cylinder (7 total)

- Argument (Gauss's Law) 4
- Magnitude 2
- Direction 1

Sphere (5 total)

- Argument 2
- Magnitude 2
- Direction 1

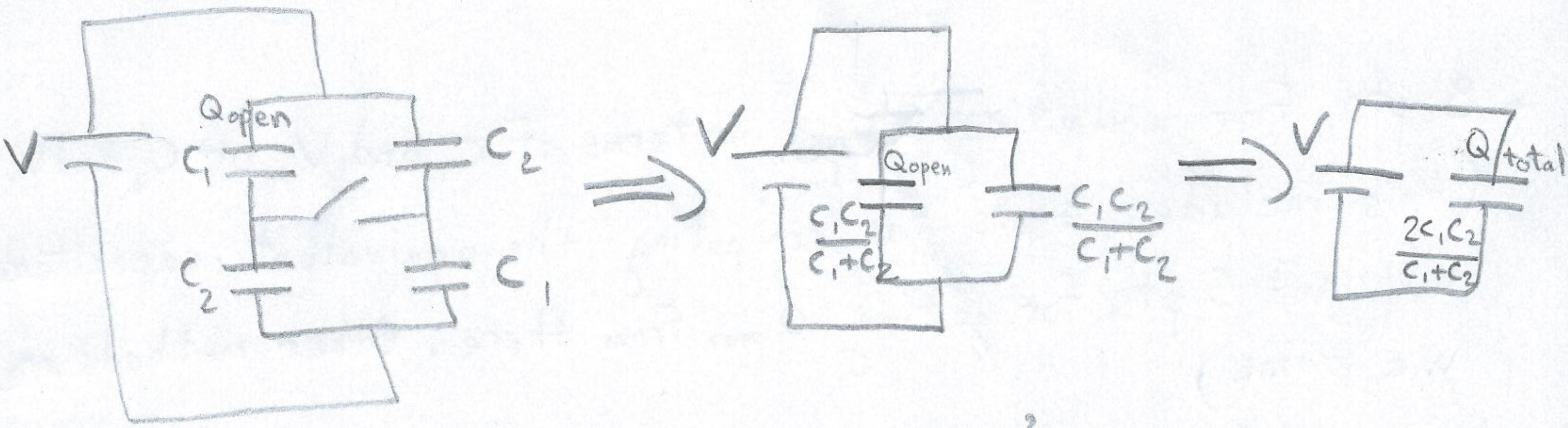
Superposition (5 total)

- Statement/use of principle: 3
- Execution: 2

Final answers (3 total)

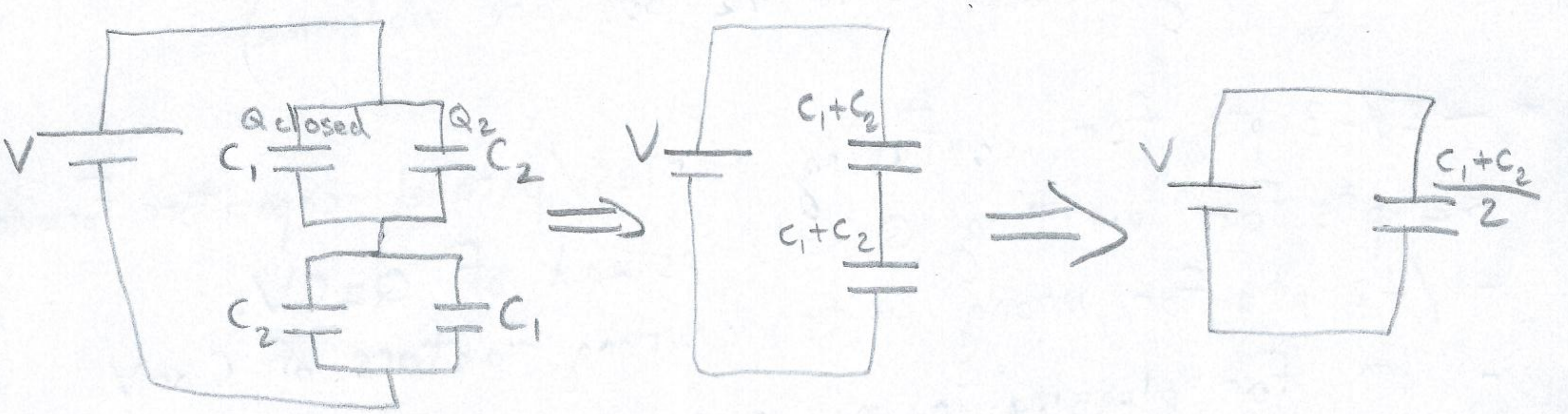
- A 1
- B 1
- C 1

Problem 2



$$Q_{total} = C_{eq}V = \frac{2C_1C_2}{C_1+C_2}V = \frac{2\cdot 5C_2^2}{6C_2}V = \frac{5}{3}C_2V$$

$$Q_{open} = \frac{1}{2}Q_{total} = \frac{5}{6}C_2V$$



By symmetry, we can "guess" $Q_{closed} = C_1\frac{V}{2}$ from the first circuit since $V/2$ must be the voltage across each capacitor.

Alternatively, proceed as before:

$$Q_{total} = C_{eq}V = \frac{C_1+C_2}{2}V = 3C_2V$$

Also $\begin{cases} Q_{total} = Q_{closed} + Q_2 \\ Q_{closed}/C_1 = Q_2/C_2 \end{cases}$. Since we just found $Q_{total} = 3C_2V$, we can solve for Q_{closed} and find:

$$Q_{closed} = \frac{5}{2}C_2V$$

So $\boxed{Q_{open}/Q_{closed} = 1/3}$

Rubric for Problem 2

- 9 pts for computing Q_{open} in terms of C_1 and V or C_2 and V. (this includes 4 pts for computing the equivalent capacitance C_{eq} and 5 pts for getting Q_{open} from there. Other methods are welcome)

- 11 pts for computing Q_{closed} in terms of C_1 and V or C_2 and V, and computing Q_{open}/Q_{closed}
(breakdown is similar to before: 4 pts for C_{eq}, and 7 pts for getting Q_{closed}. Other methods are welcome)

- −2/−3 pts for confusing series/parallel capacitors formula.
  −2 pts for writing $QC=V$ instead of $Q=CV$.
  −1/−2 pts for wrong units or wrong factors of C or V.
  −1 pt for plugging in mistake
  −4 pts for writing $Q_{open}=C_{eq}V$
  −4 pts for writing $Q_{closed}=C_{eq}V$
  } −7 pts only if both mistakes were made
  −4/−5 pts for getting a circuit wrong conceptually

5. (a) Approach: $\rho \to \vec{E} \to V$

Consider a gaussian surface which is a sphere of radius r. $\boxed{1\text{ pt}}$

Notice that the charge distribution is spherically symmetric, so $\vec{E}(\vec{r}) = E(r)\,\hat{r}$. $\boxed{1\text{ pt}}$

Then the flux through this surface is

$$\oiint_S \vec{E}\cdot d\vec{A} = \oiint_S E(r)\,\overset{1}{\cancel{\hat{r}\cdot\hat{r}}}\,dA = E(r)\oiint_S dA = E(r)\cdot 4\pi r^2 \quad \boxed{2\text{ pts}}$$

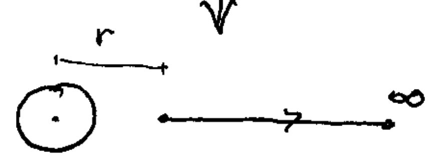
Next, we find

$$Q_{in} = \begin{cases} Q_1 + Q_2 = Q_1/2 & r > R \quad \boxed{1\text{ pt}} \\ Q_2 \cdot \left(\frac{r}{R}\right)^3 = -\frac{1}{2} Q_1 \left(\frac{r}{R}\right)^3 & r < R \quad \boxed{1\text{ pt}} \end{cases}$$

$$\oiint \vec{E}\cdot d\vec{A} = Q_{in}/\varepsilon_0$$

$$\Rightarrow E(r) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q_1/2}{r^2} \leftarrow \boxed{1\text{pt}} & r > R \\ \frac{1}{4\pi\varepsilon_0}\,\frac{Q_1\cdot\left(-\frac{1}{2}\right)}{R^2}\cdot\frac{r}{R} \leftarrow \boxed{1\text{pt}} & r < R \end{cases}$$

Use this path for the integral



Now we integrate $\vec{E}$ to get V. $\Delta V = -\int \vec{E}\cdot d\vec{\ell}$

$\boxed{r > R}$

$$\cancel{V(\infty)}^{\,0} - V(r) = -\int_r^{\infty} \frac{1}{4\pi\varepsilon_0}\,\frac{Q_1/2}{r^2}\,\hat{r}\cdot\hat{r}\,dr = -\frac{1}{4\pi\varepsilon_0}\,\frac{Q_1}{2}\cdot -\frac{1}{r}\Big]_r^{\infty}$$

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\,\frac{Q_1/2}{r} \quad \boxed{2\text{ pts}}$$

$\underline{r < R}$

$\boxed{1\,pt}$ [redacted] (Noticing there are 2 parts to the integral)

$$V(\cancel{\infty}^{\,0}) - V(r) = \underbrace{V(\infty) - V(R)}_{-\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q_1/2}{R}} + \underbrace{V(R) - V(r)}_{-\int_r^R \frac{1}{4\pi\varepsilon_0}\cdot\frac{-Q_1/2}{R^2}\cdot\frac{r}{R}\,\hat{r}\cdot\hat{r}\,dr}$$

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1/2}{R} + \left(\frac{-Q_1/2}{R^2}\cdot\underbrace{\left.\frac{1}{2}\frac{r^2}{R}\right]_r^R}_{\frac{1}{2}\left(R-\frac{r^2}{R}\right)}\right)\right]$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R}\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{4}\frac{r^2}{R^2}\right] = \frac{Q_1}{4\pi\varepsilon_0 R}\left[\frac{1}{4} + \frac{1}{4}\frac{r^2}{R^2}\right]$$

[redacted] $\boxed{2\,pts}$

(b) Charge comes ~~to rest at the~~ center after $\boxed{1\,pt}$ being released at $r=a$ $\Rightarrow$ $V(r=a) = V(r=0)$ [redacted]

because $E_i = \frac{1}{2}\cdot m\cdot 0^2 + q\cdot V(a)$
$E_f = \frac{1}{2}m\cdot 0^2 + q\cdot V(0)$ & $E_i = E_f$ [redacted] $\boxed{4\,pts}$

$$\frac{1}{4\pi\varepsilon_0}\frac{Q_1/2}{a} = \frac{Q_1}{4\pi\varepsilon_0 R}\cdot\frac{1}{4} \Rightarrow \boxed{a = 2R} \leftarrow \boxed{2\,pts}$$

Note: If you got the correct answer w/o the correct potentials, you won't get credit for the correct answer