Physics 137A — Quantum Mechanics — Spring 2013
Midterm II - Solutions

1 Simple Harmonic Oscillator [30 Points]

Consider the quantum simple harmonic oscillator (SHO) with frequency ω. The normalized eigenvectors of the SHO will be designated $|n\rangle$ or as $\psi_n(x)$. Answer the following questions about this system. You can treat each part of this problem as a separate question.

(a) Is the raising operator, $\hat{a}_+$, an observable? Explain briefly.

(b) If the current state of the system is $|\Psi(t)\rangle = c_1|1\rangle + c_3|3\rangle + c_4|4\rangle$, where c_i are complex numbers, what is $\hat{a}_-|\Psi(t)\rangle$?

(c) If the current state of the system is $|\Psi(t)\rangle = \frac{i}{4}|0\rangle + \frac{\sqrt{12}}{4}|3\rangle + \frac{\sqrt{3}}{4}|4\rangle$, what is the probability of measuring the energy to be $\hbar\omega/2$.

(d) Compute the commutator of the number operator $\hat{N} = \hat{a}_+\hat{a}_-$ with the raising operator: $\left[\hat{N}, \hat{a}_+\right]$.

(e) Sketch the first three energy eigenfunctions of the SHO.

(f) If the energy of a particular oscillator is measured and found to be $E = (2\text{ Hz})\cdot\hbar$, what are the possible values of ω for this system?

Solutions

(a) The raising operator is not Hermitian, so it is not an observable.
(b) We just need to use the rule that $\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$.

$$\begin{aligned}
\hat{a}_-|\Psi(t)\rangle &= \hat{a}_-\left(c_1|1\rangle + c_3|3\rangle + c_4|4\rangle\right) \\
&= c_1\hat{a}_-|1\rangle + c_3\hat{a}_-|3\rangle + c_4\hat{a}_-|4\rangle \\
&= c_1\sqrt{1}\,|0\rangle + c_3\sqrt{3}\,|2\rangle + c_4\sqrt{4}\,|3\rangle\,.
\end{aligned}$$

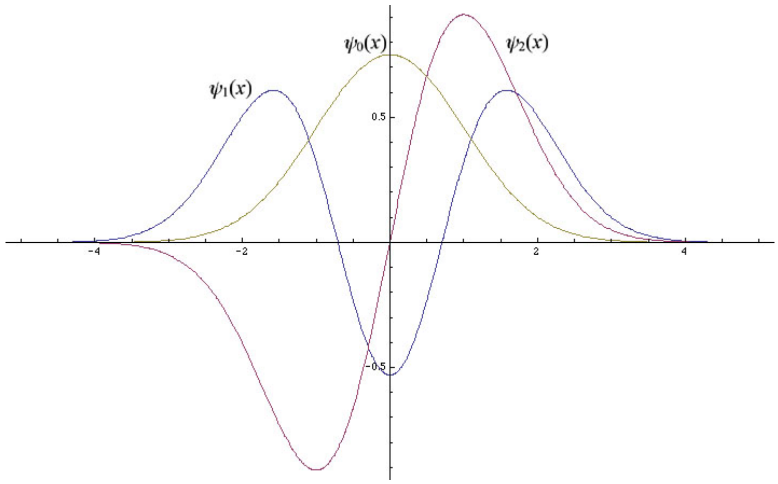
(c) The energy eigenstate associated with the energy $\hbar\omega/2$ is the ground state, $|0\rangle$. So according to our postulate, to get the probability of that measurement result, we need to take the inner product of the state with the eigenstate corresponding to this result: $P(\hbar\omega/2) = |\langle 0|\Psi(t)\rangle|^2$. Plugging in our state we find

$$\begin{aligned}
P(\hbar\omega/2) &= \left|\langle 0\,|\,\Psi(t)\rangle\right|^2 \\
&= \left|\frac{i}{4}\langle 0\,|\,0\rangle + \frac{\sqrt{12}}{4}\langle 0\,|\,3\rangle + \frac{\sqrt{3}}{4}\langle 0\,|\,4\rangle\right|^2 \\
&= \left|\frac{i}{4}\right|^2 \\
&= \frac{1}{16}.
\end{aligned}$$

So there is a 1 in 16 chance that the stated energy would be measured.
(d) To do this part we need to recall that the commutator of the ladder operators is $[\hat{a}_-, \hat{a}_+] = 1$. Another way of saying this is $\hat{a}_-\hat{a}_+ = 1 + \hat{a}_+\hat{a}_-$. Now we compute the value in question:

$$\begin{aligned}
\left[\hat{N}, \hat{a}_+\right] &= \left[\hat{a}_+\hat{a}_-, \hat{a}_+\right] \\
&= \hat{a}_+\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_+\hat{a}_- \\
&= \hat{a}_+\left(1 + \hat{a}_+\hat{a}_-\right) - \hat{a}_+\hat{a}_+\hat{a}_- \\
&= \hat{a}_+.
\end{aligned}$$

(e) Here is a 'sketch' made with the help of Mathematica...



(f) If the frequency is known, the allowed values of an energy measurement for the SHO are just the usual $E_n = \hbar\omega(n + 1/2)$. So what we have is one of these energies, we do not know which. So we will get a different possible ω for each value that n could take. So we have

$$\begin{aligned} \hbar\omega(n + 1/2) &= (2\text{ Hz}) \cdot \hbar \\ \omega(n + 1/2) &= 2\text{ Hz} \\ \omega &= \frac{2\text{ Hz}}{n + 1/2}. \end{aligned}$$

2 A Fewtrinos [30 Points]

In this problem we will see a simple Quantum Mechanical model of neutrino oscillations. Consider a two state system with an orthonormal basis $|1\rangle$ and $|2\rangle$. These two vectors are eigenvectors of the 'flavor' operator, with $|1\rangle$ representing an electron neutrino and $|2\rangle$ representing a muon neutrino.

(a) If neutrinos are massless, the Hamiltonian for a system like this would have been:

$$\hat{H} = E\,|1\rangle\,\langle 1| + E\,|2\rangle\,\langle 2|\,,$$

where E is a positive real constant. If the neutrino initially started as an electron neutrino, $|\Psi(0)\rangle = |1\rangle$, what is the state of the system at later times?

(b) If the flavor of the resulting state at time $t = T$ is measured, what is the probability that you would measure the neutrino to be a muon neutrino?

(c) If instead the neutrinos have masses, the Hamiltonian is given by the following:

$$\hat{H} = E\,|1\rangle\,\langle 1| + E\,|2\rangle\,\langle 2| + \epsilon\,|1\rangle\,\langle 2| + \epsilon\,|2\rangle\,\langle 1|\,.$$

Here ϵ is some real constant value that depends on the masses of the two flavors of neutrino. What are the energy eigenvalues and eigenvectors of this Hamiltonian?

(d) If a neutrino subject to the Hamiltonian in part (c) begins as an electron neutrino at time $t = 0$, $|\Psi(0)\rangle = |1\rangle$, what is the probability at time $t = T$ that you would measure the neutrino to be a muon neutrino?

Solutions

(a) The given Hamiltonian is diagonal in the flavor basis. That is, the two kets $|1\rangle$ and $|2\rangle$ are eigenvectors of this Hamiltonian, each with eigenvalue E. So the evolution of this system is just a matter of expanding the initial condition in the energy eigenbasis, and then tacking on the exponential time phase factor. The initial condition is equal to one of the energy eigenvectors, so we only have to add the phase factor.

$$|\Psi(t)\rangle = e^{-iEt/\hbar}\,|1\rangle\,.$$

(b) The probability, according to our postulates, is given by $P(\text{muon}) = |\,\langle 2|\Psi(T)\rangle\,|^2$. So we compute this inner product to find

$$\langle 2|\Psi(T)\rangle = e^{-iET/\hbar}\,\langle 2|1\rangle = 0.$$

So there is no chance that the electron neutrino would be measured to be a muon neutrino. This is true at any time.

(c) There are a number of ways to answer this question, but I will present the method using a matrix representation. The matrix representation of the Hamiltonian is just the collection of numbers $H_{i,j} = \langle i|\hat{H}|j\rangle$. That is, the matrix has the number $H_{i,j}$ in its i-th row and j-th column. So we have, working through the four products:

$$\mathbf{H} = \begin{pmatrix} E & \epsilon \\ \epsilon & E \end{pmatrix}.$$

To find the eigenvalues, we develop the characteristic equation, which is just the determinant of the previous matrix with λ subtracted off the diagonal, set equal to zero. I find

$$(E-\lambda)^2 - \epsilon^2 = 0.$$

This has the solution $\lambda = E \pm \epsilon$. So the eigenvalues of this Hamiltonian are different from the Hamiltonian in part (a), and that difference is given by ϵ.

Now for the eigenvectors. We want to solve the following equation:

$$\begin{pmatrix} E & \epsilon \\ \epsilon & E \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (E+\epsilon)\begin{pmatrix} a \\ b \end{pmatrix}.$$

In this expression, a and b are complex numbers that give the matrix representation of the eigenvector in the $|1\rangle$, $|2\rangle$ basis. So we have something like $|+\rangle = a\,|1\rangle + b\,|2\rangle$. The top row of this equation gives

$$Ea + \epsilon b = (E+\epsilon)a.$$

Or $a = b$. So if we want a normalized eigenvector, we have to take $a = b = 1/\sqrt{2}$. This means that the plus eigenvector (the eigenvector for the plus energy) is $|+\rangle = 1/\sqrt{2}\,|1\rangle + 1/\sqrt{2}\,|2\rangle$.

To get the other eigenvector, we can run the calculation again using the other eigenvalue. Also, we can recognize that because the Hamiltonian represents an observable quantity, we know it must be Hermitian. It is also clear that it is Hermitian from the expression in the problem statement. This means that we can find an orthonormal basis of eigenvectors for this Hamiltonian. So, we can find a vector orthogonal to the previous. So we require $\langle -|+\rangle = 0$. The only vector (up to scaling) that satisfies this is $|-\rangle = 1/\sqrt{2}\,|1\rangle - 1/\sqrt{2}\,|2\rangle$. You can test this with the Hamiltonian and see that it is indeed the expected eigenvector.

(d) The solution strategy here is to once again apply postulate 3:

$$P(\text{muon}) = \left|\langle 2|\Psi(T)\rangle\right|^2.$$

All we have to do is find the state of the system at the correct time and plug into the previous expression. To find the state of the system at later times, we will use our main method. First we expand the state in energy eigenvectors:

$$|\Psi(0)\rangle = |1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right).$$

Since we have already solved the energy eigenvalue problem in part (c), we can now just add on the exponential phase factors.

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i(E+\epsilon)t/\hbar}\,|+\rangle + e^{-i(E-\epsilon)t/\hbar}\,|-\rangle\right).$$

At this point we are ready to plug this into our probability formula:

$$\begin{aligned}
P(\text{muon}) &= |\langle 2|\Psi(T)\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(e^{-i(E+\epsilon)T/\hbar}\,\langle 2|+\rangle + e^{-i(E-\epsilon)T/\hbar}\,\langle 2|+\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(e^{-i(E+\epsilon)T/\hbar}\frac{1}{\sqrt{2}} - e^{-i(E-\epsilon)T/\hbar}\frac{1}{\sqrt{2}}\right)\right|^2 \\
&= \left|\frac{1}{2}\left(e^{-i(E+\epsilon)T/\hbar} - e^{-i(E-\epsilon)T/\hbar}\right)\right|^2 \\
&= \left|\frac{1}{2}e^{-iET/\hbar}\left(e^{-i\epsilon T/\hbar} - e^{i\epsilon T/\hbar}\right)\right|^2 \\
&= \left|\frac{1}{2}e^{-iET/\hbar}(-2i)\sin\left(\frac{\epsilon T}{\hbar}\right)\right|^2 \\
&= \left|\frac{1}{2}e^{-iET/\hbar}(-2i)\right|^2\left|\sin\left(\frac{\epsilon T}{\hbar}\right)\right|^2 \\
&= \sin^2\left(\frac{\epsilon T}{\hbar}\right).
\end{aligned}$$

So if there is this cross term, then the electron neutrino will transform into muon neutrinos in flight. And this actually happens! Neutrinos out in the universe behave just like this: the energy eigenstates for the neutrinos are not flavor eigenstates, so we see neutrino oscillations.