Problem 1

(a) The linear charge density, λ, can be found by integrating the charge density, ρ over the cross-sectional area of the cylinder. In this case,

$$\lambda = \int_A \rho(\mathbf{x}) d^2 \mathbf{x} = \int_0^a \rho(r) \cdot 2\pi r dr = \int_0^a 2\pi \rho_0 (r - \frac{r^3}{a^2}) dr = \frac{\pi \rho_0 a^2}{2}. \quad (1)$$

(b) Use a cylindrical gaussian surface of unit height to solve for the electric field. By symmetry, we know that the electric field must point in the radial direction. Using this setup we find that,

$$\mathbf{E} = E_r \hat{r} = \frac{Q_{encl}(r)}{2\pi \epsilon_0 r}; \ Q_{encl}(r) = \left\{ \begin{array}{ll} \pi \rho_0 (r^2 - \frac{r^4}{2a^2}) & \text{if } r < a \\ \lambda & \text{if } r \geq a. \end{array} \right. \quad (2)$$

(c) You can find the potential by integrating the electric field along a radial path. The electric potential is,

$$\begin{array}{ll} V(r) & = -\int_0^r E_r(r) dr + V(r = 0) \\ & = \left\{ \begin{array}{ll} -\frac{\rho_0}{2\epsilon_0} (\frac{r^2}{2} - \frac{r^4}{6a^2}) & \text{if } r < a \\ -\frac{\lambda}{2\pi \epsilon_0} \log \frac{r}{a} - \frac{\rho_0 a^2}{6\epsilon_0} & \text{if } r \geq a. \end{array} \right. \end{array} \quad (3)$$

(d) The total energy per unit length can be found by using the expression for energy density $u = \frac{1}{2}\epsilon_0 |\mathbf{E}|^2$ of the electric field. In this case, the linear energy density is, as in part (a), the energy density integrated over the area of interest,

$$\int_0^R \epsilon_0 |\mathbf{E}|^2 \pi r dr = \frac{\pi \rho_0^2}{2\epsilon_0} \int_0^a r(r - \frac{r^3}{2a^2})^2 dr + \frac{\lambda^2}{4\pi \epsilon_0} \log (R/a) \quad (4)$$

It's left as an exercise to the students and grader to explicitly write out this integral's solution. It's not hard, but it is messy!

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MT2 | Problem 1 | Lec1 | SP13

Rubric: Each part has a total of 5pt, which then breaks down as following:

a) 3pt: Setting up the correct integral with ALL variables and parameters involved.
   2pt: The correct final answer.

b) 1pt: To write the Gauss' law: $\oint \vec{E} \cdot d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_0}$. No credit for figure-only.

c) 1pt: To write the integral equation of electric potential and field: $-\int \vec{E} \cdot d\vec{r} = V_{ba}$

   1pt: To set up the correct integral with all variables and parameter involved for r<a
   1pt: The correct final answer for r<a
   1pt: To set up the correct integral with all variables and parameter involved for r>a
   1pt: The correct final answer for r>a

d) 2pt: To write down the equation for energy density of the electric field
   2pt: To set up the correct integral and breaking it down to two integrals
   1pt: The final answer

Problem 2

(a) Use a cylindrical gaussian surface that has small radius and pierces one side of the capacitor to find that the contribution to find that the electric field must be given by $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{z} = \frac{Q}{\epsilon_0 L^2}\hat{z}$. Note that this is because the other endcap of the surface is in the conductor and so has $\mathbf{E} = 0$. In this case, we know that the charge is related to the voltage as,

$$V = \int_\gamma \mathbf{E} \cdot d\mathbf{x} = \frac{Qd}{\epsilon_0 L^2} \to C = \epsilon_0 L^2/d.$$

(b) You can find he voltage drop by finding the new capacitance when the dielectric is inserted. Treat the capacitor as two capacitors in parallel, one with a dielectric and area $L \cdot s$ and one without the dielectric and area $L \cdot (L - s)$. Therefore,

$$C = \frac{\epsilon_0(L - s)L}{d} + \frac{\kappa\epsilon_0 sL}{d} \to V = \frac{Qd}{\epsilon_0 L}\left(\frac{1}{\kappa s + (L - s)}\right)$$

(c) I guess we already solved for this in (b), didn't we?

(d) Energy stored in the capacitor is given as $U = \frac{QV(s)}{2}$, where $V(s)$ is given as in (b).

(e) The force on the dielectric is given as,

$$\mathbf{F} = -\nabla U = \frac{Q^2 d}{\epsilon_0 L}\frac{\kappa - 1}{((\kappa - 1)s + L)^2}\hat{s}.$$

Note that this means that the dielectric, if released, would slide into the capacitor. Cool, huh?

a) Gauss' Law (+1):

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_o} \Rightarrow \vec{E} = -\frac{\sigma}{\varepsilon_o}\hat{z} = -\frac{Q}{\varepsilon_o L^2}\hat{z}$$ inside the capacitor and zero elsewhere

$$\qquad\qquad\qquad\qquad\qquad\qquad \text{-1/2 if E is quoted directly}$$

$$\Delta V = -\int_{-s}^{(+)}\vec{E}\cdot d\vec{\ell} = \frac{Qd}{\varepsilon_o L^2} \quad (+1)$$

$$Q = CV \Rightarrow C = \frac{\varepsilon_o L^2}{d} \qquad (+2) \qquad\qquad \text{-1 for algebraic mistake}$$

b) $C_{left} = \frac{\varepsilon_o L(L-s)}{d}$ (+1) $\quad C_{right} = \frac{K\varepsilon_o Ls}{d}$ (+1) , where $\quad 0 \le s \le L$

Two capacitor in parallel: $\quad C = C_{left} + C_{right}$ (+1)

$$= \frac{\varepsilon_o L}{d}(L+(K+1)s) \text{ (+1), -1 for algebraic mistake}$$

c)

$$Q = CV \quad (+2)$$

$$V = \frac{Qd}{\varepsilon_o L[L+(K-1)s]} \quad (+2), \text{ -1 for algebraic mistake}$$

Note: some students misinterpreted this question as the difference of the potential before and after the dielectric is inserted. However if this was true, the question should read, "the drop of the potential difference across the capacitor before and after the dielectric is inserted", or something like this. When we say voltage drop ACROSS the plates, the adjective "across" has fully determined what we mean. Lastly, whenever there's such confusion during the exam, it does no harm to raise up your hand and ask for clarification.

d) $\mathcal{E} = \frac{1}{2}\frac{Q^2}{C}$ (+2)

$$= \frac{1}{2}\frac{Q^2 d}{\varepsilon_o L[L+(K-1)s]} \text{ (+2), -1 for algebraic mistake}$$

e) $\vec{F} = -\vec{\nabla}\mathcal{E}$ (+1)

$$= \hat{s}\frac{d}{ds}\frac{1}{2}\frac{Q^2 d}{\varepsilon_o L[L+(K-1)s]}$$

$$= -\frac{Q^2 d}{2\varepsilon_o L}\hat{s}\left\{-[L+(K-1)s]^{-2}(K-1)\right\} \text{ (+2) for derivation}$$

$$= \frac{(K-1)Q^2 d}{2\varepsilon_o L[L+(K-1)s]^2}\hat{s} \text{ (+1)}$$

remember

$$0 \le s \le L \text{ , and the direction of increasing s is going into the capcitor.}$$

Problem 3

(a) The drift velocity can be thought up as the average velocity that ions travel in the medium when an electric field is applied. Ions are accelerated by the electric field until they hit another particle in the medium with frequency $1/\tau_\pm$. The collision time is chosen such that the momentum exchange over this period would be equivalent to a head-on collision and elastic with a particle of equal mass, so this would reset the ion's velocity to zero and the process would reset. The average velocity will be equivalent to the drift velocity, assuming constant acceleration. Recall that $\mathbf{v} = \mathbf{p}/m$ and $\mathbf{F} = d\mathbf{p}/dt$, so for a particle of charge q and mass m, $v(t) = \frac{qE}{m}t$ along $\mathbf{E}$. Therefore, The drift velocity may be given as,

$$\mathbf{v_{D\pm}} = \mp \frac{eE}{2m_\pm}\tau_\pm \hat{r},$$

where $\hat{r}$ points radially away from the earth. Note the sign!

(b) Current density is given by $\mathbf{J} = \sum q_i n_i \mathbf{v_{Di}}$. In this case, considering both the positive and negative ions, and using drifts from part (a), we have,

$$\mathbf{J} = en_+\mathbf{v_{D+}} - en_-\mathbf{v_{D-}}.$$

(c) Recall that the resistivity, ρ, is defined via Ohm's Law: $\mathbf{J} = \mathbf{E}/\rho$. In this case, you can directly use your solution from (b) to find that,

$$\rho = \frac{2}{e^2}\left(\frac{n_+\tau_+}{m_+} + \frac{n_-\tau_-}{m_-}\right)^{-1}$$

(d) From this, we can get the power per unit volume dissipated through collisions as,

$$du/dt|_{\text{col}} = -|\mathbf{J}|^2\rho = -|\mathbf{J}||\mathbf{E}| = -\frac{e^2E^2}{2}\left(\frac{n_+\tau_+}{m_+} + \frac{n_-\tau_-}{m_-}\right).$$

Quick note: depending on the rigor of your derivation, you might not have the factor of $\frac{1}{2}$ in your solution. This is due to the defintion of "collision time" being somewhat nebulous.

Problem 3

PART A (4pts)
1pt) V=p/m,
1pt) F=dq/dt
1pt) v=qEt/m
1pt) final answer

PART B (4pts)

2pts) J=sum(qnVd)
2pts) final answer
1pt lost if the "J"s were not added up.

PART C (6pts)

2pts) J=E/rho
4pts) Substituting the correct variables in and getting the final answer
2pts lost if there were incorrect algebraic operations as well as any extra variables.

Part D (6pts)

2pts) du/dt= -J^2rho =
2pts) du/dt =-J*E
2pts) final answer
2pts were taken away if the final answer was not in terms of the proper variables
and for incorrect algebraic operations.

An additional 2 points were removed if one type of charged particle was not taken
into consideration in the calculations.

Problem 4

a) Refer to the attached photocopy.

b) The flux of the electric field through the cylinder can be written as

$$\Phi_E = \oiint_{cylinder} \vec{E} \cdot d\vec{A} = \iint_{lateral} \vec{E}_l \cdot d\vec{A} + \iint_{top} \vec{E}_t \cdot d\vec{A} + \iint_{bottom} \vec{E}_b \cdot d\vec{A}$$

(i) For the lateral surface,

$$\vec{E}_l = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2+R^2}\hat{r}$$

$$\Phi_{lateral} = \iint_{lateral} \vec{E}_l \cdot d\vec{A} \rightarrow \int_{-\infty}^{+\infty} \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2+R^2}\frac{R}{\sqrt{z^2+R^2}} 2\pi R dz$$

$$= \frac{qR^2}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{dz}{(z^2+R^2)^{\frac{3}{2}}} = \frac{qR^2}{2\pi\varepsilon_0}\frac{2}{R^2}$$

$$= \frac{q}{\varepsilon_0}$$

We have taken the limit that the cylinder is infinitely long, i.e. L>>R.

(ii) For the lid at each end, the magnitude of the electric field vanishes to the 0th order when we take the cylinder to be infinitely long, while the surface area remains as finite. Therefore, the fluxes at the top and bottom lids of the cylinder could be considered to be both zero.

Therefore,

$$\boldsymbol{\Phi_E = \frac{q}{\varepsilon_0}}$$

which verifies the Gauss's law.

c) According to Gauss's law, we expect the answer obtained in part (b) to be equal to the flux of the electric field through a spherical Gaussian surface.

Notes (not included in the solutions; reference only):

For a spherical Gaussian surface of radius r,

$$\Phi_E^{(Sphere)} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}4\pi r^2 = \frac{q}{\varepsilon_0} = \Phi_E^{(Cylinder)}$$

i.e., as we expected, the electric flux generated by a charge sitting at the "geometrical center" through a cylindrical surface is indeed equal to the flux through a spherical surface.
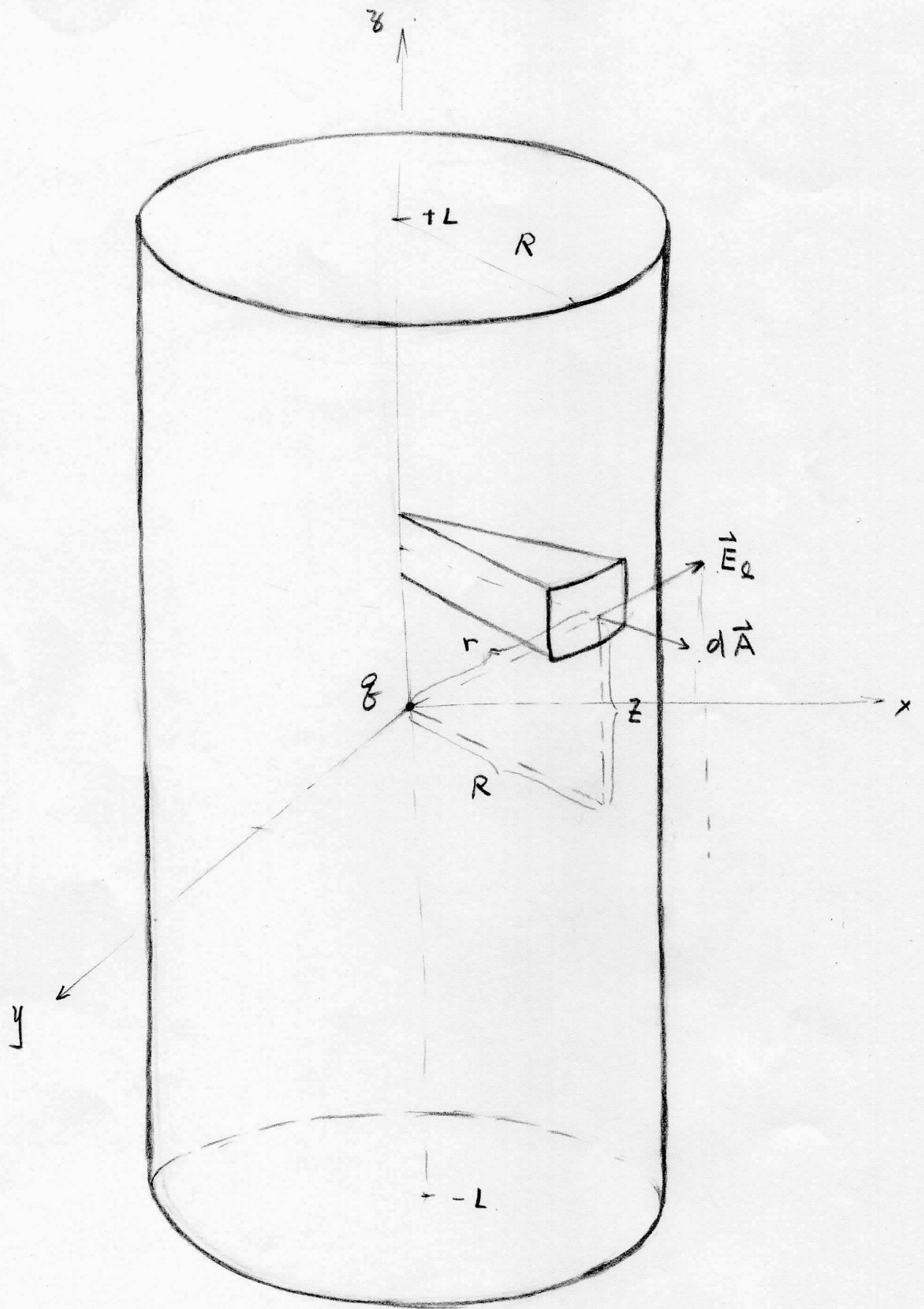
Problem 4 Rubric

1 Part a - 2 pts

2 was awarded if a picture clearly showed a cylinder with a charge in the center, otherwise 0

2 Part b - 14 pts

Here there were two general methods for solving. The intended one was for the flux to be calculated directly by doing an integral, not by using Gauss' Law. The point breakdown in this case was:

- 4 pts for demonstrating conceptual understanding of what the problem is asking. Anyone who found electric field as an integral of something with dq in it lost these points. These points were also lost is the flux was written as $E_{\text{tot}} \cdot A_{\text{tot}}$. A small number of people received 2 pts here, but mostly it was 0 or 4. One way to get 2 points subtracted was to write something like $dE = \frac{kdq}{r^2}$, which doesn't make sense here, and then to go ahead and solve the rest of the problem correctly.

- 2 pts for having $E = \frac{kq}{r^2}\hat{r}$ somewhere on the page

- 3 pts for parametrizing the surface correctly, either $dA = 2\pi r dl$ (where dl could be dx, dz, etc) or $dA = r dl d\theta$. The latter case also requires doing a θ integral from 0 to 2π. Many people received partial credit here.

- 1 pt for both noticing and correctly evaluating the $\cos(\theta)$ or $\sin(\theta)$ factor associated with the dot product of electric field and the normal vector. Whether it should be a cosine or a sine depends on how you set up the angle. This was a very commmon mistake: people drew the angle for which it should be $\sin(\theta)$, but found $\cos(\theta)$ instead, since that is what we find most of the time. Also note that even if you correctly evaluated it, if it appeared for explicitly the wrong reason, you could not get this point. (In retrospect, I should have made this be two points, one for noticing and one for evaluating, but oh well.)

- 2 pts for evaluation - most people who got everything above got both of these points, and most people who didn't really know what to do above did not get them. Students who did basically the right thing beforehand could lose these points if they ended up with a vector at the end ($\hat{r}$ still in the answer), if they integrated only from 0 to ∞, if they missed a factor of 2, etc

- 2 pts for "lids"/"ends" - 1 pt for mentioning them, 1 for correctly concluding that they make no contribution (I did not care whether or not the explanation was valid). In a way, these two points are really in part c. If you calculate the flux through the whole surface, then that actually answers part b. On the other hand, to say that the flux through this cylinder is the same as for a sphere (part c), you need to talk about the fact that an infinitely long cylinder is not technically a closed surface (by the definition we're using in this class, at least). The points were moved to part b so that part c could be graded the same way regardless of the method of solving part b.

- -3 pts for wrong units - the above points add to the total, 14. Independent of all of these, any student who had the wrong units in some expression somewhere in part b lost 3 points, down to a minimum of 0 total points for part b. Check your units! (By the way, about 3/4 of the class lost these points.)

The second solution method is just to say that by Gauss' Law, the flux should be $\frac{q}{\epsilon_0}$. That was worth 9 points. The remaining 5 points were given for stating that the ends of the cylinder don't contribute, since that is necessary to conclude something with Gauss' Law. The observation that the ends don't contribute is worth more points with this solution method than the previous one because it is a larger fraction of the work needed to solve the problem. (Here the points are really part of part b, which is why I moved the points for the other solution method).

3 Part c - 4 pts

Here 2 points were given for saying that they *should be* the same by Gauss' Law (this would have required a mention of the ends/lids of the cylinder, but that was moved to b). Students who said they should not be the same because of the ends of the cylinder got no points here, but got 1 in part b. To get these 2 points, however, you had to have the right answer $\frac{q}{\epsilon_0}$ for either the cylinder or the sphere, to show that you understand the reasoning and did not just succeed at a 50% probability guess.

2 more points were given for having the correct value for the sphere. These points were given if the answer to part b was correct and the student said the results should be the same, or if the flux was correctly calculated for the sphere independently.

Many, many people, some of whom knew what it meant to calculate flux in part b, still found electric field in part c. This received 0 points, even if the correct value for flux was present on a line in this calculation.

4 Other comments

A few notes for any students who actually read this:

- You don't get points by magically changing from $(R^2 + x^2)^{-1}$ to $(R^2 + x^2)^{-3/2}$. In fact, you lose an additional 3 points for having the wrong units. If your expression does not match the given integral, you should take it as a sign that you have forgotten about some actual physics in the problem.

- If you have $\frac{x}{(R^2+x^2)^{3/2}}$, you cannot integrate it and get the same result you would get by integrating $\frac{1}{(R^2+x^2)^{3/2}}$. Lots of people did this. In fact, with the x on top it is an odd function and gives 0.

- Factors of R tended to appear and disappear randomly throughout the problem. This always resulted in -3 points for units, even if the final answer had the correct units.

Problem 5

a) $V = 0$ and $\vec{E} = 0$ according to the symmetries of the charge configuration.

b) We evaluate the assembling energy by calculating the required work to move the charges from the infinity to the required positions one by one.

(i) Charge Q located at (-L, -L)
$$W_{-L,-L} = 0$$

(ii) Charge -Q located at (L, -L)
$$W_{L,-L} = -\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2L}$$

(iii) Charge Q located at (L, L)
$$W_{L,L} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2\sqrt{2}L} - \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2L}$$

(iv) Charge -Q located at (-L, L)
$$W_{-L,L} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2\sqrt{2}L} - \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{L}$$

The total work required to assemble such a system is
$$W = W_{-L,-L} + W_{L,-L} + W_{L,L} + W_{-L,L}$$

Therefore,
$$\boldsymbol{W = -\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{L}\frac{4-\sqrt{2}}{2}}$$

c) According to the symmetry of the system, the electric field should point along the direction (L, L), applying the coordinate system indicated in the diagram. For the calculations below, the positive direction of the electric field is understood to be along the vector (L, L).

(i) Electric field due to the charge Q located at (-L, -L)
$$E_{-L,-L} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\left(\sqrt{2}L+s\right)^2}$$

(ii) Electric field due to the charge Q located at (L, L)
$$E_{L,L} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\left(\sqrt{2}L-s\right)^2}$$

(iii) Electric field due to the charge –Q's located at (L, -L) and (-L, L)
$$E_{-Q} = -\frac{1}{4\pi\varepsilon_0}\frac{2Q}{\left(\sqrt{2}L\right)^2+s^2}\frac{s}{\sqrt{\left(\sqrt{2}L\right)^2+s^2}} = -\frac{1}{4\pi\varepsilon_0}\frac{2Qs}{\left[\left(\sqrt{2}L\right)^2+s^2\right]^{\frac{3}{2}}}$$

Therefore, the total electric field is
$$E = E_{-L,-L} + E_{L,L} + E_{-Q}$$

Hence,

$$E = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left(\sqrt{2}L + s\right)^2} - \frac{1}{\left(\sqrt{2}L - s\right)^2} - \frac{2s}{[2L^2 + s^2]^{\frac{3}{2}}}\right]$$

The direction is along the vector (L, L).

a) Taking s<<L into account, the above expression for E can be Taylor expanded. We keep the first order term in the following procedure.

$$E = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{2L^2}\frac{1}{\left(1+\frac{s}{\sqrt{2}L}\right)^2} - \frac{1}{2L^2}\frac{1}{\left(1-\frac{s}{\sqrt{2}L}\right)^2} - \frac{2s}{2\sqrt{2}L^3}\frac{1}{\left[1+\left(\frac{s}{\sqrt{2}L}\right)^2\right]^{\frac{3}{2}}}\right]$$

$$\approx \frac{Q}{8\pi\varepsilon_0 L^2}\left[\left(1 - \frac{\sqrt{2}s}{L}\right) - \left(1 + \frac{\sqrt{2}s}{L}\right) - \frac{\sqrt{2}s}{L}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{3\sqrt{2}}{2}\frac{Q}{L^3}s$$

Thus the magnitude of the electric field is

$$\frac{1}{4\pi\varepsilon_0}\frac{3\sqrt{2}}{2}\frac{Q}{L^3}s$$

The direction is along the vector (-L, -L).

Therefore, to the first order approximation, the electric force experienced by the charge q is

$$F = -\frac{1}{4\pi\varepsilon_0}\frac{3\sqrt{2}}{2}\frac{Qq}{L^3}s$$

The charge is expected to exhibit harmonic oscillation, with frequency

$$\omega = \sqrt{\frac{k}{m}}$$

with

$$k = \frac{3\sqrt{2}}{2}\frac{1}{4\pi\varepsilon_0}\frac{Qq}{L^3}$$

Therefore,

$$\omega = \sqrt{\frac{3\sqrt{2}Qq}{8\pi\varepsilon_0 L^3 m}}$$

Problem 5 Rubrics

a) (4 points) 2 credits for potential and 2 credits for electric field. For each part, 1 point for correct result and 1 point for correct interpretation. No credit for wrong interpretations. No credit for wrong results.

b) (6 points) 4 points in total for moving charges one by one to the required position (approximately 1 point for each step listed above). 1 point for adding the work done for each step. 1 point for the final answer.
For adding interacting energies between charges directly, 4 points for correct formulation and 2 points for the correct result.

c) (5 points) 3 points for calculating electric field due to each charge (approximately 1 point for each step listed above). 2 points for the final answer (1 point for correct direction).

d) (5 points) 2 points for Taylor expansion (1 point for trying). 2 points for the correct formula of frequency (ω=Sqrt[k/m]. 1 point for stating the force's linear dependence on s.) 1 point for the correct frequency.