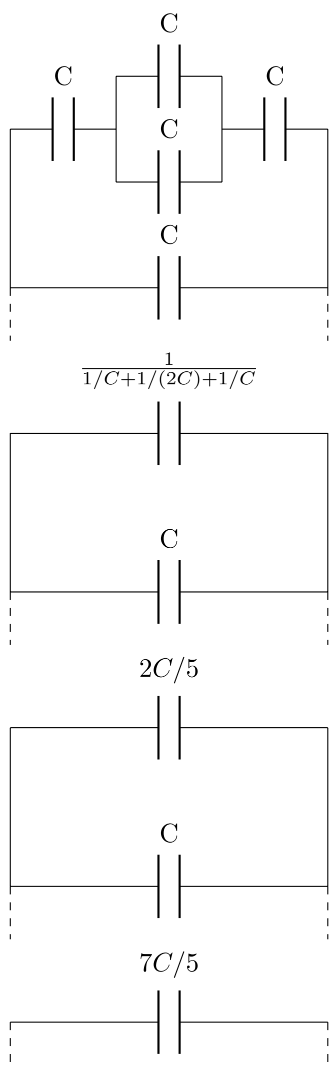
Problem 1

(1) The insulator acts like a dielectric. We know that this effectively modifies the electric field by a constant. We can absorb this into the permitivity, $\varepsilon_0 \to \varepsilon$. The electric field in a capacitor is $\frac{\sigma}{\varepsilon}$. So, $\frac{E_D}{E_0} = \frac{\varepsilon_0}{\varepsilon} = 1/K$. It turns out that this result is generally true in an arbitrary capacitor, it need not be parallel plates.

(2) The capacitance is given by integrating the electric field across the capacitor and realizing that $Q = CV$. Thus, $V = \int \vec{E} \cdot d\vec{\ell} = \int_0^d \frac{\sigma}{\varepsilon_0} dx = \frac{Qd}{A\varepsilon_0}$ and it follows that $C = \frac{A\varepsilon_0}{d}$. So it is clear that $\frac{C}{C_0} = K$.

(3) Here is shown the decomposition of adding capacitors in each step we use the fact that the addition of capacitors in parallel is $C_1 + C_2$ and in series is $(C_1^{-1} + C_2^{-1})^{-1}$.



Problem 2

(1) To answer this question we need to think about the speed at which a signal travels in a medium. The electron that leaves the switch when you throw it does not get to the light before the light turns on. This follows by the simple speed argument described in the problem statement.

The proper way to think about the current is that when we close the circuit, there is an electric field established in the circuit. This electric field travels (is established) at nearly the speed of light. Once it is set up, the electrons in the wire at all points will begin to move. This means that in order to have light we have to wait for the E-field to be established at the position of the electrons near the light bulb. This happens at nearly the speed of light.

Note that it cannot be actually instantaneous. Suppose that I turn a switch on in New York and this activates a light in Berkeley. If we could turn it on faster than the time required to send light over that distance, we could send information faster than the speed of light. This violates special relativity.[1]

(2) The drift velocity is very small compared to the velocity of light because the electrons bump into atoms which causes multiple disruptions in their trajectory (scattering phenomena) and therefore significantly decreases the distance traveled per unit time along the direction of the electric field (parallel to the wire).

(3) Since temperature allows the atoms to move further away and faster from their equilibrium position, the probability of electron-atom collisions (microscopic origin of resistivity) increases with temperature, which in turn increases the resistivity.

[1]Its also unlikely because of the horrible condition of Amrerican internet speeds.

Problem 2

Because of the nature of this problem, I graded this question “holistically”. I made an effort to understand your intentions and grade accordingly.

(a) was worth 7 points in total. These were divided as follows:

- 6 points was for mentioning the electric field as the force that drove the movement of electrons.
- 1 point was for mentioning the speed at which the electric field travels. This was an essential part of a full answer, because it actually addresses the question of why lightbulbs turn on almost instantaneously. However, I was very generous in giving this point to anyone whose answer even implied near instantaneous propagation of the electric field.

A correct analogy would receive full points if it addressed the problem of propagation speed. The water-out-of-a-tap analogy was very popular but somewhat misleading, but since it is in your book I ignored this if you offered an otherwise correct physical explanation for current. Arguments that involved electrostatic, non-collision based explanations but were not correct were awarded between 2 and 5 points, depending on how close it was to the actual physics. Arguments based on collisions as the mode of energy transmissions received no points.

(b) was worth 5 points in total, of which all are essentially awarded for anything mentioning electron collisions with atoms. Partial (3 point) credit was awarded for mentioning the random movement of electrons, without specifying collisions as the reason for the random movement. Arguments that were derived from collisions but did not explain their origin in terms of collisions received 2 out of 5 points. Arguing from equations like $j = nqv_D$ was awarded no points, because knowing this equation does not reflect *physical* understanding of why drift velocity is small.

(c) was worth 8 points in total.

- 3 points was for pointing out that higher temperature means that the thermal objects are moving/vibrating faster.
- 3 more points was for arguing that higher temperature led to more collisions.
- 2 points was from arguing from this that the resistance increases with temperature. One would receive these points from arguing from the book’s formula for the temperature dependence on resistivity AND stating that α is positive (usually). However, a full answer would explain this formula as in the first two bullets.

Explanations from thermal expansion are incorrect but were awarded between 3 and 5 points, for at least offering a physical explanation. For thermal expansion, points were given depending on the conclusions one drew from the thermal expansion (i.e. did you correctly look at the expansion of A and ℓ, is your logic self-consistent, do you know what resistivity means, etc.)

Since the atom is neutral, the total charge outside the nucleus must be -q. (Note for a sphere $V = \frac{4}{3}\pi r^3$ so $dV = 4\pi r^2 dr$):

$$-q = \int \rho(r)dV = \int_R^\infty \frac{A}{r^5}4\pi r^2 dr = 4\pi A \int_R^\infty \frac{1}{r^3}dr = \frac{4\pi A}{2R^2} = \frac{2\pi A}{R^2} \tag{1}$$

$$A = \frac{-qR^2}{2\pi} \tag{2}$$

Now we use Gauss's law to calculate the field. We use a spherical Gaussian surface of radius $r > R$. Because the charge distribution is radial (and we are not concerned about the distribution of charge for $r < R$ since Gauss's law tells us only the charge inside matters) Gauss's law tells us that the electric field must be constant along this surface and can be taken out of the flux integral. Further the symmetry tells us that the field lines should point radially, and are thus parallel to $d\vec{A}$.

$$\int \vec{E} \cdot d\vec{A} = E(r)4\pi r^2 \tag{3}$$

We also have (where V_{GS} is the volume contained in the Gaussian surface):

$$\frac{q_{in}}{\epsilon_0} = \frac{1}{\epsilon_0}(q + \int \rho(\eta)dV_{GS}) = \frac{1}{\epsilon_0}(q - \frac{qR^2}{2\pi}\int_R^r \frac{4\pi\eta^2}{\eta^5}d\eta) = \frac{q}{\epsilon_0}(1 - 2R^2(\frac{1}{2R^2} - \frac{1}{2r^2})) = \frac{qR^2}{\epsilon_0 r^2} \tag{4}$$

Equation 3 and 4 are equal by Gauss's law, so:

$$\vec{E} = \frac{qR^2}{4\pi\epsilon_0 r^4}\hat{r} \tag{5}$$

We will use $V(\infty) = 0$. I will integrate along any straight path that goes from ∞ to the r we desire (this works because lines of constant r are equipotentials):

$$V(r) = V(r) - V(\infty) = -\int_\infty^r \vec{E} \cdot d\vec{l} = \int_r^\infty \frac{qR^2}{4\pi\epsilon_0\eta^4}d\eta = \frac{qR^2}{12\pi\epsilon_0 r^3} \tag{6}$$

Problem 3 Rubric :

The atom is charge neutral :

$$-q = \int \rho\ (r)\ dV = \int_R^\infty \frac{A}{r^5} * 4 * \pi * r^2\, dr = \frac{2 * \pi * A}{R^2}$$

$$A = \frac{-R^2 * q}{2 * \pi}$$

If recognize charge neutral, get 3 points. If set the integral right, get 1 point. If get the right answer, get 1 point. 5 points in total.

Then use Gauss ' s Law to calculate the E field. Treat the nucleus as a point charge. The entire system (nucleus and electrons) enjoys a spherical symmetry. The E field lines point radially.

$$\int E\, dA = E * 4 * \pi * r^2$$

If recognize using Gauss ' s law, get 1 point. If mentioning spherical symmetry, get 1 point. 2 points in total.

$$\frac{q_{in}}{\epsilon_0} = \frac{1}{\epsilon_0}\left(q + \int \rho\ (\eta)\ dV_{GS}\right) = \frac{1}{\epsilon_0}\left(q - \frac{R^2 * q}{2 * \pi}\int_R^r \frac{4 * \pi * \eta^2}{\eta^5}\, d\eta\right) = \frac{q * R^2}{\epsilon_0\, r^2}$$

If recognize to count both the contribution from nucleus and electrons, get 4 points. Setting up the correct integral, get 1 point. Get the correct answer, get 1 point. 6 points in total.

$E = \dfrac{qR^2}{4\,\pi\epsilon_0\, r^4}$, **direction : point outward radially.**

Get the right magnitude, 1 point. Get the correct direction, 1 point. 2 points in total.

$$V\ (r) = V\ (r) - V\ (\infty) = -\int_\infty^r E\, dl = \int_r^\infty \frac{qR^2}{4\,\pi\epsilon_0\, \eta^4}\, d\eta = \frac{qR^2}{12\,\pi\epsilon_0\, r^3}$$

If recognize $V\ (\infty) = 0$ and integral E field to get V, get 2 points. Set up the right integral, 1 point. 1 point for path description. We integral along any straight path from ∞ to the r. 1 point for correct final answer. 5 points in total.

Problem 4

1. The electric field ends up pointing in the $\theta = 180$ direction. Students interpreted the axes differently, but this should have been the end result. The following is for the case where $\theta = 0$ corresponds to the x-axis.

   (2 pts) The charge distribution has symmetry about the xz plane as $\cos\theta = \cos -\theta$ so $E_y = 0$.

   (2 pts) If we pair the electric field generated by a chunk at θ which has charge $\lambda_0 \cos\theta$ and a chunk at $\theta + \pi$ which has charge $\lambda_0 \cos\theta + \pi = -\lambda_0 \cos\theta$, we see that $E_z = 0$. The only component left is E_x.

   (2 pts) It is easy to then see that as the positive charge is on the right-half plane and the negative charge is on the left half plane that the electric field points in the $-\hat{x}$ direction.

   Note that solutions with unclear explanations were awarded less points.

2. Recall that to calculate the electric field, we need to calculate $\int \frac{dq\hat{r}}{4\pi\epsilon_0 r^2}$ where $\vec{r}$ is the vector from a chunk of charge to the point at which we are trying to calculate the electric field. $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$.

   (2 pts) If we use polar coordinates, $\vec{r} = (-R\cos\theta, -R\sin\theta, z)$. In order to find the x component of the field, we need to take the x component of $\hat{r}$, which is $\frac{-R\cos\theta}{\sqrt{R^2+z^2}}$. One point was given if you included a $\cos\theta$ in the x component, but didn't have the whole expression correct.

   Therefore, at $(0, 0, z)$, we have

   $$dE_x = \int_0^{2\pi} d\theta \frac{1}{4\pi\epsilon_0} \frac{\lambda_0 R\cos\theta}{R^2+z^2} \frac{-R\cos\theta}{\sqrt{R^2+z^2}} = \frac{\lambda_0}{4\pi\epsilon_0} \frac{-R^2}{(R^2+z^2)^{\frac{3}{2}}} \int_0^{2\pi} d\theta \cos^2\theta \tag{1}$$

   (1 pt) for having the correct expression for $dQ = \lambda_0 \cos\theta R d\theta$.

   It follows that the electric field at $(0, 0, z)$ is

   $$\vec{E} = \pi \frac{\lambda_0}{4\pi\epsilon_0} \frac{R^2}{(R^2+z^2)^{\frac{3}{2}}} (-\hat{x}) \tag{2}$$

   (2 pts) for correct answer.

3. The electric field of the second ring is found from the previous one by making the replacements

   (1 pt)$\lambda_0 \to -\lambda_0$ and

   (1 pt)$z \to z - 2R$.

   Then adding this with the previous equation gives us

   $$\vec{E} = \frac{1}{2}\frac{\lambda_0}{4\pi\epsilon_0}\left(\frac{R^2}{(R^2+z^2)^{\frac{3}{2}}} - \frac{R^2}{(R^2+(z-2R)^2)^{\frac{3}{2}}}\right)(-\hat{x}) \tag{3}$$

   (1 pt) correct magnitude

   (1 pt) for giving your answer as a vector

4. You could plot either E or E_x. Also it was acceptable to either give $z \in [0, 2R]$ or $z \in (-\infty, \infty)$.

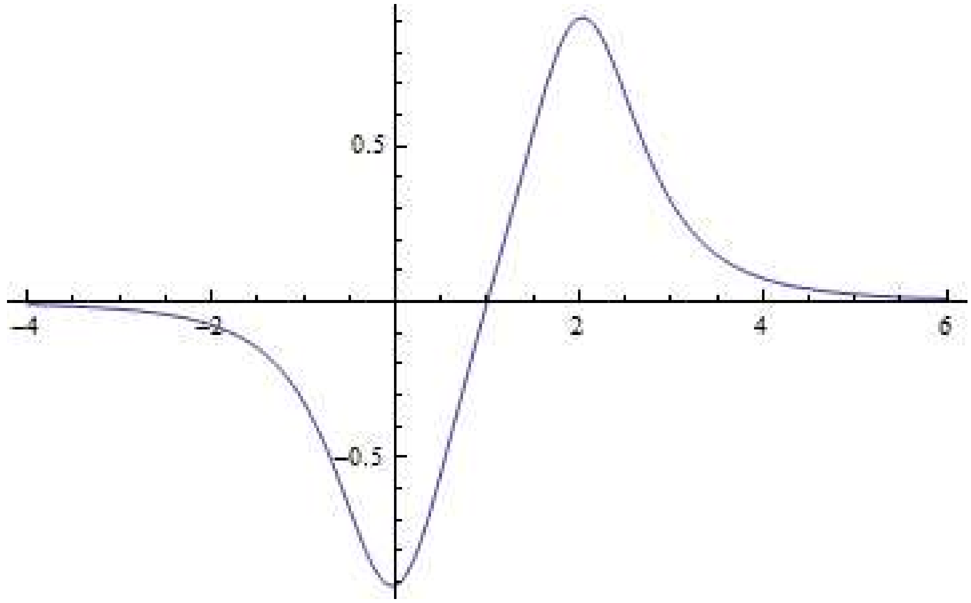
   NOTE: If you got an answer of $E = 0$ in part (b) had you had the correct limits $\theta \in (0, 2\pi)$, you did not receive credit for the correct plot. So basically, if you had just $\int \cos\theta$, which should be zero, then you did not get points for the graph.

   (1 pt) Labeling axes.

   (2 pts) $E_x = 0$ at $z = R$,

   (2 pts) $E_x > 0$ for $z > R$, and $E_x < 0$ for $z < R$.

   Note: the field drops off as $|z|$ becomes large.


0.5
-0.5
-4
-2
2
4
6

Problem 5

(1) The full potential on the x-axis is found by summing the potential for all of the charges. Each charge contributes $\frac{kQ}{r}$ to the full potential. Here we assume without loss of generality that $V(\vec{r} \to \infty) = 0$. The full potential is:

$$V(x) = \frac{-kq}{|x+3a|} + \frac{3kq}{|x+a|} + \frac{-3kq}{|x-a|} + \frac{kq}{|x-3a|}$$

(Of course this is valid at all points except at the location of any charge.)

(2) Since we are interested in $x \to \infty$ we can assume WLOG that $x > 3a$. Thus we can dispense with the absolute value signs:

$$V(x) = \frac{-kq}{x+3a} + \frac{3kq}{x+a} + \frac{-3kq}{x-a} + \frac{kq}{x-3a} = \frac{48kqa^3}{(x+3a)(x-3a)(x+a)(x-a)} = \frac{48kqa^3}{x^4 - 10a^2x^2 + 9a^4}$$

Now we can take the asymptotic expansion. You could have used the formula given or noticed that x^4 is the leading order. We get:

$$V(x) \longrightarrow \frac{48kqa^3}{x^4}$$

(Note that the units are correct.)

(3) We apply the relation $\vec{E} = -\nabla(V)$. We only need the x-component because V only has x dependance.

$$\vec{E} = -\frac{\partial}{\partial x}\left(\frac{48kqa^3}{x^4}\right)\hat{x} = \frac{192kqa^3}{x^5}\hat{x}$$

We can use the potential along the x axis to find the field precisely because we know that the field will point along $\hat{x}$, by the rotational symmetry of the charge distribution. More explicitly, in the expression:

$$\vec{E} = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

we already know that the last two components are 0 by smmetry and we can choose to calculate the potential along the only direction where the gradient is non-zero.

(4) You can see by inspection that the potential vanishes identically along the y-axis. At any point on the axis it gets exactly opposite contributions from charges that are equal distance away, $\pm 3q$ and $\pm 1q$.

The electric field however is not zero on this axis. This is clear because the potential is positive to the left of the axis and negative to the right. Therefore, it's derivative must be non-zero at the axis because it's continuous. (This follows from the mean value theorem.) In addition it must point to the right.

More explicitly suppose that move off of the y-axis from $(0, y)$ to $(-\epsilon, y)$ for some small ϵ. Then at this point V has a larger contribution from the $+3q$ than the $-3q$, and although it has a larger contribution from $-q$ than q that difference is smaller than the previous one because the lengths are longer. Therefore $V(-\epsilon, y) > 0$. A similar argument shows that $V(\epsilon, y) < 0$.