MIDTERM 2, BORDEL SOLUTIONS PART I

1. Problem 1 - Linear Charge Distribution

1.1. **a.** There can be no electric field component at C along the y-axis, as the system is left invariant upon inversion across the x-axis. There is also no component in or out of the page, as the system is confined to the plane of the page. Therefore, the field at C must be entirely along the x-direction. As the charge distribution above C is entirely positive, the electric field must be pointing down the x-direction, away from those charges. Note that this argument holds for both subsystems as well.

To calculate the electric field at C, we will calculate electric fields of the semi-circle and the rod separately and combine the expressions by the principle of superposition.

1.2. **b.** Because the electric field is entirely along the x-direction:

$$\vec{E} = \left(\vec{E} \cdot \hat{x}\right) \hat{x} \tag{1}$$

Calculating the magnitude at C due to the semi-circle charge:

$$\begin{aligned} \vec{E} \cdot \hat{x} &= \int_{SC} \frac{k_e \lambda dl}{r^2 (l)} \hat{r}(l) \cdot \hat{x} \\ &= \int_{-\pi/2}^{\pi/2} \frac{k_e \lambda R d\theta}{R^2} \cos(\theta) \\ &= \frac{k_e \lambda}{R} \int_{-\pi/2}^{\pi/2} d\theta \cos(\theta) \end{aligned} \tag{2}$$

where I have directed $\theta = 0$ along $-\hat{x}$.

Evaluating the integral:

$$\vec{E} \cdot \hat{x} = \frac{k_e \lambda}{R} \left[\sin(\theta)\right]_{-\pi/2}^{\pi/2} = \frac{2 k_e \lambda}{R} \tag{3}$$

and so

$$\vec{E} = \frac{2 k_e \lambda}{R} \hat{x} \tag{4}$$

1.3. **c.** Again, calculating the magnitude of the electric field at C due to the line charge:

$$\begin{aligned} \vec{E} \cdot \hat{x} &= \int_{-L}^{L} \frac{k_e \lambda dl}{r^2(l)} \hat{r}(l) \cdot \hat{x} \\ &= \int_{-L}^{L} \frac{k_e \lambda dl}{R^2 + l^2} \cos(\theta) \end{aligned} \tag{5}$$

defining θ as before. We can evaluate the cosine in terms of other quantities:

$$\begin{aligned} \vec{E} \cdot \hat{x} &= \int_{-L}^{L} \frac{k_e \lambda dl}{R^2 + l^2} \frac{R}{\sqrt{R^2 + l^2}} \\ &= k_e \lambda R \int_{-L}^{L} \frac{dl}{(R^2 + l^2)^{3/2}} \end{aligned} \tag{6}$$

Evaluating the integral using the Reminder:

$$\begin{aligned} \vec{E} \cdot \hat{x} &= k_e \lambda R \left[\frac{l}{R^2 \sqrt{R^2 + l^2}} \right]_{-L}^{L} \\ &= \frac{2k_e \lambda}{R} \frac{L}{\sqrt{R^2 + L^2}} \end{aligned} \tag{7}$$

and

$$\vec{E} = \frac{2k_e \lambda}{R} \frac{L}{\sqrt{R^2 + L^2}} \hat{x} \tag{8}$$

2. Problem 2 - Surface Charge Distribution

2.1. **a.** For each region we apply Gauss's law using a finite (length L) cylindrical surface, as the electric field is cylindrical radial.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0} \tag{9}$$

Using a cylinder of radius $r < R_1$, we have the result

$$E 2\pi r L = \frac{0}{\epsilon_0} \rightarrow \vec{E} = \vec{0} \text{ for } r < R_1 \tag{10}$$

Using a cylinder of radius $R_1 < r < R_2$, we have the result

$$E 2\pi r L = \frac{\beta L}{\epsilon_0} \rightarrow \vec{E} = \frac{\beta}{2\pi \epsilon_0 r} \hat{r} \text{ for } R_1 < r < R_2 \tag{11}$$

For a cylinder of radius $r > R_2$, we have the result

$$E2\pi rL = \frac{\beta L - \beta L}{\epsilon_0} = 0 \to \vec{E} = \vec{0} \text{ for } r > R_2 \tag{12}$$

2.2. **b.** We will set the potential at infinity to zero. Integrating the null electric field to any point outside of both cylinders shows us that the potential everywhere outside the cylinder is zero.

$$V_{III}(r) = 0 \tag{13}$$

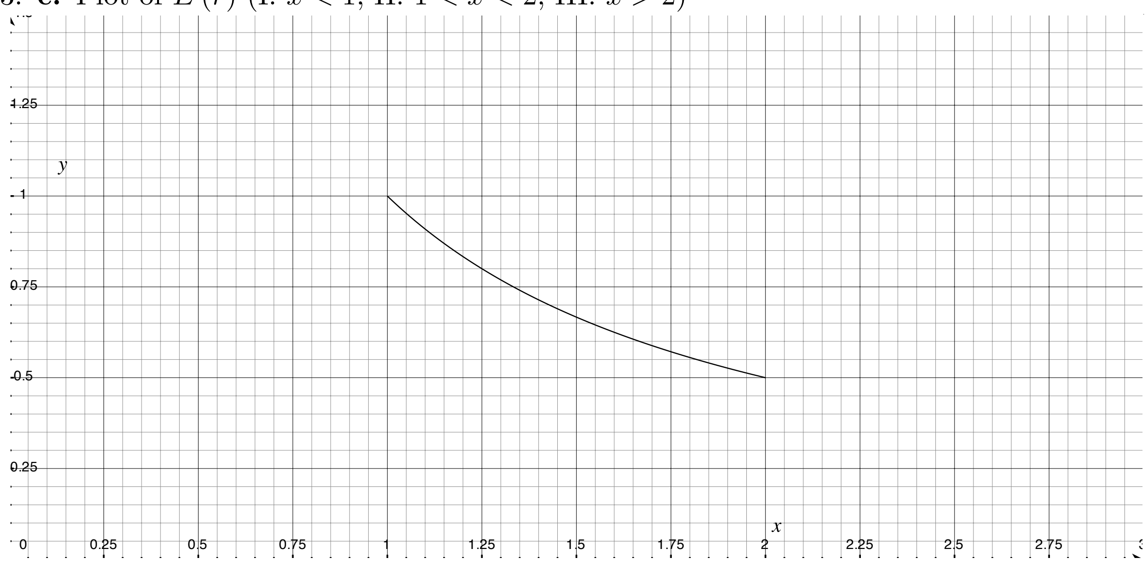
The electric field inside the outer cylinder and outside the inner cylinder is non-zero, so the potential is changing:

$$\begin{aligned} V_{II}(r) &= 0 - \int_{R_2}^{r} \vec{E} \cdot d\vec{x}' \\ &= -\int_{R_2}^{r} \frac{\beta dr'}{2\pi\epsilon_0 r'} \\ &= \frac{\beta}{2\pi\epsilon_0} \ln\left(\frac{R_2}{r}\right) \end{aligned} \tag{14}$$

The electric field inside both cylinders is zero, and so the potential in this region is constant:

$$V_I(r) = V_{II}(R_1) = \frac{\beta}{2\pi\epsilon_0} \ln\left(\frac{R_2}{R_1}\right) \tag{15}$$

2.3. **c.** Plot of $E(r)$ (I: $x < 1$, II: $1 < x < 2$, III: $x > 2$)



Plot of $V(r)$ (I: $x<1$, II: $1<x<2$, III: $x>2$)

2.4. **d.** For any subsection of the finite charged cylinders of length L, the total charge on each shell is $\pm Q = \pm\beta L$. The potential difference across the shells is $\Delta V = \frac{\beta}{2\pi\epsilon_0}\ln\left(\frac{R_2}{R_1}\right)$. Using our formula for capacitance:

$$C = \frac{Q}{\Delta V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)} \tag{16}$$

We see that the capacitance of this subsection is proportional to its length. The capacitance per unit length of this capacitor is

$$c = \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{R_2}{R_1}\right)} \tag{17}$$

3. Problem 3 - Charged Spheres in Equilibrium

3.1. **a.** By Gauss's law, the electric field outside of an isolated spherical conductor of radius R_a and charge Q_a is the same as that of a point charge:

$$\vec{E} = \frac{Q_a}{4\pi\epsilon_0 r^2}\hat{r} \tag{18}$$

and so, in this region, the potential is that of a point charge as well:

$$V = \frac{Q_a}{4\pi\epsilon_0 r} \tag{19}$$

at the surface:

$$V_s = \frac{Q_a}{4\pi\epsilon_0 R_a} \tag{20}$$

and at some radius r far away:

$$V_{far} = \frac{Q_a}{4\pi\epsilon_0 r} \tag{21}$$

3.2. **b.** Because the spheres are separated by a distance much greater than their radii, the potential of one is fairly uniform in the region of the other. Therefore,

$$\begin{aligned} V_1\left(Q_1, Q_2\right) &= \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 d} \\ V_2\left(Q_1, Q_2\right) &= \frac{Q_2}{4\pi\epsilon_0 R_2} + \frac{Q_1}{4\pi\epsilon_0 d} \end{aligned} \tag{22}$$

3.3. **c.** Using the formula for potential energy of a charge distribution:

$$U = \frac{1}{2}\sum_i Q_i V\left(x_i\right) \tag{23}$$

we have:

$$\begin{aligned} U &= \frac{1}{2}Q_1 V_1\left(Q_1, Q_2\right) + \frac{1}{2}Q_2 V_2\left(Q_1, Q_2\right) \\ &= \frac{Q_1^2}{8\pi\epsilon_0 R_1} + \frac{Q_1 Q_2}{4\pi\epsilon_0 d} + \frac{Q_2^2}{8\pi\epsilon_0 R_2} \end{aligned} \tag{24}$$

3.4. **d.** Substituting in x:

$$U\left(x\right) = \left(x^2\right)\frac{Q_0^2}{8\pi\epsilon_0 R_1} + \left(x - x^2\right)\frac{Q_0^2}{4\pi\epsilon_0 d} + \left(x^2 - 2x + 1\right)\frac{Q_0^2}{8\pi\epsilon_0 R_2} \tag{25}$$

Setting the first derivative to zero to find a potential minimum:

$$\begin{aligned} 0 &= \left(2x\right)\frac{Q_0^2}{8\pi\epsilon_0 R_1} + \left(1 - 2x\right)\frac{Q_0^2}{4\pi\epsilon_0 d} + \left(2x - 2\right)\frac{Q_0^2}{8\pi\epsilon_0 R_2} \\ -\frac{1}{d} + \frac{1}{R_2} &= \frac{x}{R_1} - \frac{2x}{d} + \frac{x}{R_2} \\ x &= \frac{\frac{1}{R_2} - \frac{1}{d}}{\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{2}{d}\right)} \end{aligned} \tag{26}$$

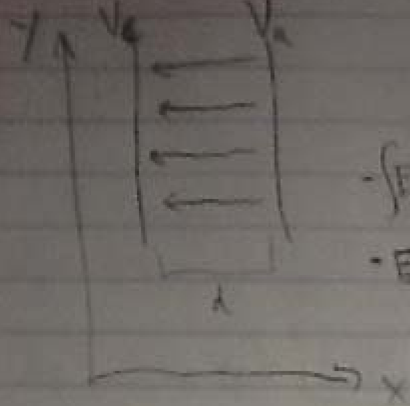
Assuming $d >> R_1, R_2$ we have:

$$x \approx \frac{1}{\left(1 + \frac{R_2}{R_1}\right)} \tag{27}$$

This could also have been obtained by only considering the first and third terms in equation (25).

4.

a)



$V_a > V_c$

$-\int^{d} \vec{E}\cdot d\vec{l} = \Delta V$

$-Ed = V_a - V_c$

$E = \frac{V_c - V_a}{d}$ $\qquad \vec{E} = \frac{V_c - V_a}{d}\hat{x}$

b)

$\vec{F}_{el} = q_e\vec{E} = q_e\frac{V_c - V_a}{d}\hat{x}$

c) $W = \int_0^d F\cdot dx = q_e(V_c - V_a)\frac{d}{d} = q_e(V_c - V_a)$

or just $W = -q_e \Delta V$

d) $\vec{a} = \frac{\vec{F}}{m_e} = \frac{(V_c - V_a)q_e}{d m_e}$

$s = v_0 t + \frac{1}{2}at^2$ → b/c $\vec{a}$ const $\quad s = d,\ v_0 = 0$

$d = \frac{1}{2}at^2$

$\sqrt{\frac{2d}{a}} = t$

$t = \sqrt{\frac{2d^2 m_e}{(V_c - V_a)q_e}}$

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a.) $C_i = \frac{A\epsilon_0}{\ell_1}$ $C_f = \frac{A\epsilon_0}{\ell_2}$

V const so $V_i = V_f = V$

$U_i = \frac{1}{2} C_i V_i^2 = \boxed{\frac{1}{2}\frac{A\epsilon_0}{\ell_1} V^2}$

$U_f = \frac{1}{2} C_f V_f^2 = \boxed{\frac{1}{2}\frac{A\epsilon_0}{\ell_2} V^2}$

$\Rightarrow \Delta U = \frac{A\epsilon_0 V^2}{2}\left[\frac{1}{\ell_2} - \frac{1}{\ell_1}\right]$

b.) C varies as a function of x (distance between plates), but V is constant b/c of battery

Since $V = \frac{Q}{C} \Rightarrow$ Q changes as a fcn of x

$Q = VC = V\frac{A\epsilon_0}{x}$

$E_{plate} = \frac{Q}{2A\epsilon_0} = \frac{V}{2x}$

$F = Q E_{plate} = \left(\frac{VA\epsilon_0}{x}\right)\cdot\left(\frac{V}{2x}\right) = \left(\frac{V^2 A\epsilon_0}{2}\right)\frac{1}{x^2}$

c.) $W = \int_{\ell_1}^{\ell_2} \vec{F}\cdot d\vec{x} = \left(\frac{V^2 A\epsilon_0}{2}\right)\int_{\ell_1}^{\ell_2}\frac{dx}{x^2} = -\left(\frac{V^2 A\epsilon_0}{2}\right)\left[\frac{1}{x}\right]_{\ell_1}^{\ell_2} = \frac{-V^2 A\epsilon_0}{2}\left[\frac{\ell_1 - \ell_2}{\ell_1 \ell_2}\right]$

d.) Work on plates $- \Delta U_{cap} +$ Work by batt $= 0$

Work by batt $= \Delta U_{cap} -$ Work on plates