1. (4 points) Write down the 3-point DFT and IDFT in matrix form. The entries of the matrices involved should be written as complex numbers in rectangular form (i.e. $a + bi$).

Solution:

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}} \\ 1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{-1}{2} + i\frac{\sqrt{3}}{2} & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \\ 1 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \\ 1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{-1}{2} - i\frac{\sqrt{3}}{2} & \frac{-1}{2} + i\frac{\sqrt{3}}{2} \\ 1 & \frac{-1}{2} + i\frac{\sqrt{3}}{2} & \frac{-1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}$$

2. (4 points) Match the following signals to their respective spectrograms. Assume all signals are of duration 1000 samples.

a) $x(n) = 1 + \cos(\frac{\pi}{2}n)$

b) $x(n) = \sin(\frac{\pi}{2}n)$

c) $x(n) = 1$

d) $x(n) = \begin{cases} 0 & : n < 500 \\ \cos(\frac{\pi}{3}n) & : n \geq 500 \end{cases}$

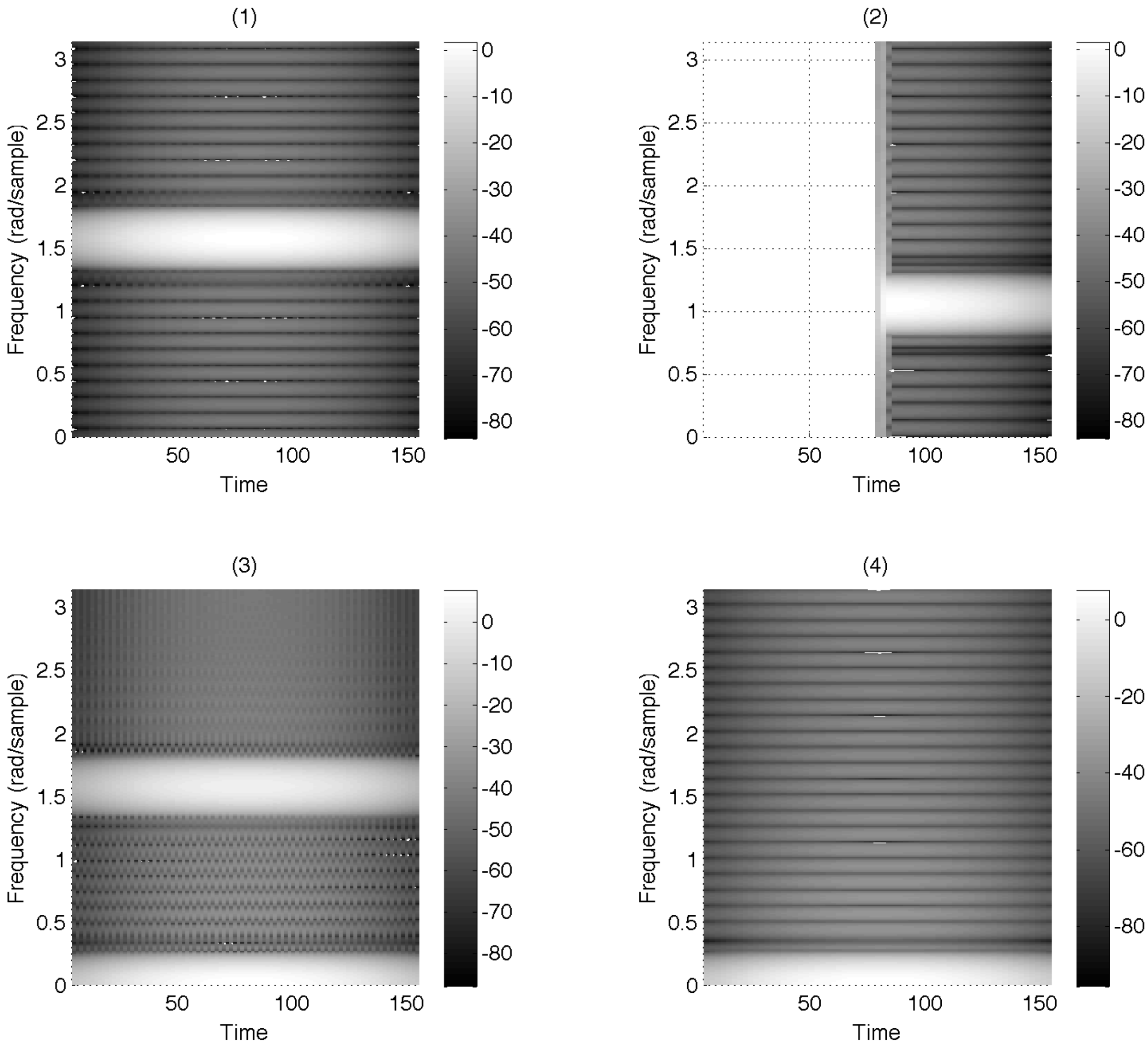
Solution:

a) (3)

b) (1)

c) (4)

d) (2)

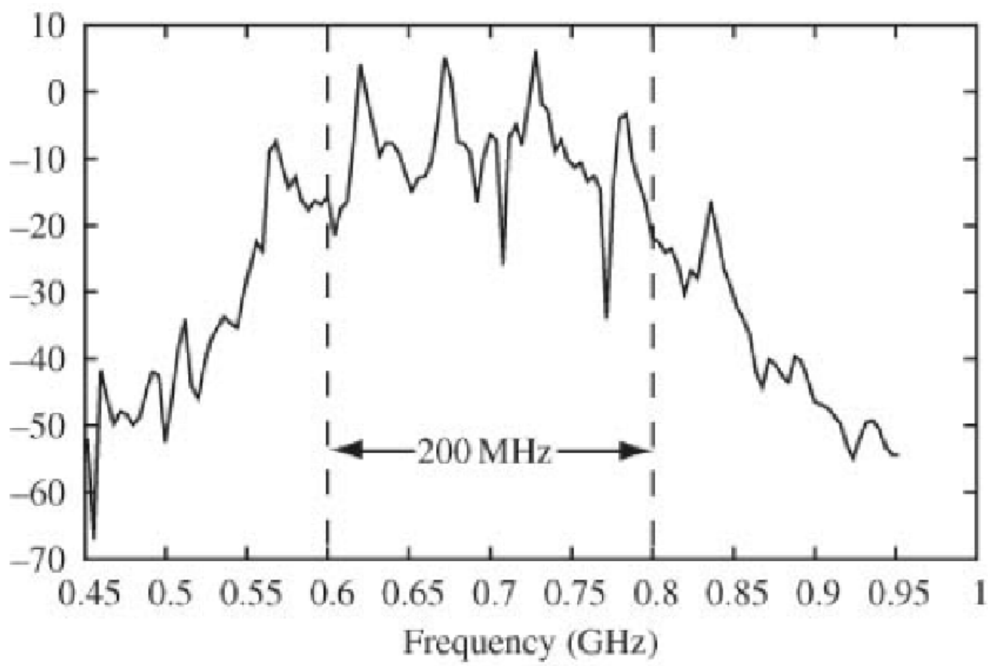


3. (5 points) Suppose the response of a discrete-time LTI system to a step input is $g(n)$ (the so-called *step response*). Here, a step input is the signal $u(n) = 0$ for $n < 0$ and $u(n) = 1$ for $n \geq 0$. Does the step response $g(n)$ fully specify what the LTI system is? If so, compute the impulse response $h(n)$ of the system in terms of $g(n)$. If not, give an example of two LTI systems having the same step response.

Solution:

The step response does specify the LTI system completely. We are given that input $u(n)$ produces output $g(n)$. By time-invariance, input $u(n-1)$ will produce output $g(n-1)$. Note that $\delta(n) = u(n) - u(n-1)$. By linearity, $\delta(n)$ will produce output $g(n) - g(n-1)$. Hence, the impulse response is $h(n) = g(n) - g(n-1)$.

4. (6 points) Consider the frequency response of a wireless channel shown below. The magnitude is plotted on the dB scale. We restrict ourselves to within the 200 MHz band.



a) (1 point) At what frequency is the channel strongest?

b) (1 point) At what frequency is the channel weakest?

c) (2 points) Roughly by what factor is the strongest channel stronger than the weakest channel? (an order-of-estimate is fine)

d) (2 points) Consider an OFDM system using 8 sub-carriers (i.e. using an 8-point DFT). What is the frequency (in Hz) of the sub-carrier having the strongest channel?

Solution:

a) Strongest at 0.72 GHz.

b) Weakest at 0.77 GHz.

c) Difference in dB between strongest and weakest channel = 5 - (-35) dB = 40 dB. Therefore, $20 \log_{10} \frac{H_{\text{strongest}}}{H_{\text{weakest}}} = 40$, so $\frac{H_{\text{strongest}}}{H_{\text{weakest}}} = 100$.

d) 25 MHz subcarrier (baseband) has the strongest signal.

5. (7 points) You want to build a system to generate music with certain frequency components . You have at your disposal:

- p-point DFT blocks, for $p = 128, 512$ and 1024.
- p-point IDFT blocks, for $p = 128, 512$ and 1024.
- D/A converters at sampling frequencies $f_s = 2.5, 10$ and 20 k-samples/s.
- A/D converters at sampling frequencies $f_s = 2.5, 10$ and 20 k-samples/s.

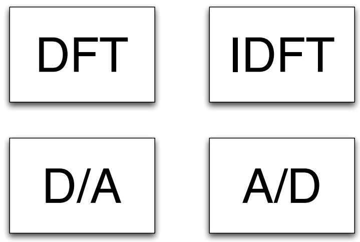
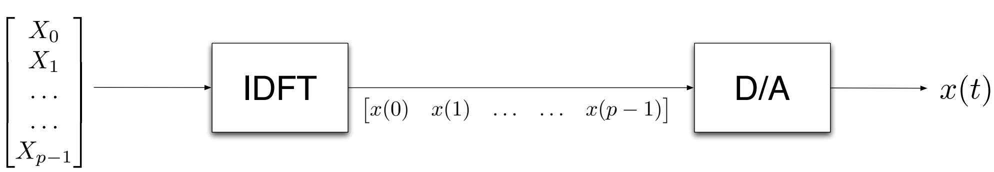


Figure 1: Available blocks

a) (2 points) Build your system by filling in each of the blank blocks in the system diagram below by one of the available blocks above.

b) (5 points) Choose values for the block size p, the sampling frequency f_s and the input to your system to generate at the output music with frequency components at approximately 880 Hz and 1760 Hz for a duration of approximately 50 ms.

Solution:



a)

b) The length of the discrete-time signal we generate has length p, so the 50 ms duration of the signal constrains us as follows:

$$\frac{p}{f_s} \approx 50 \times 10^{-3},$$

leaving us with the possible (p, f_s) pairs of $(128, 2500), (512, 10000), (1024, 20000)$.

Regardless of our choice of (p, f_s), each coefficient X_m is attributed to a frequency (in Hz) of $\frac{m}{p} \cdot f_s$. The 880 Hz and 1750 Hz are both real frequencies and therefore two coefficients, X_m, X_{-m}, must be attributed to each component.

$$880 = \frac{m_1}{p} \cdot f_s \rightarrow m_1 = 44, -m_1 = -44$$

$$1760 = \frac{m_2}{p} \cdot f_s \rightarrow m_2 = 88, -m_2 = -88$$

$\pm m_1, \pm m_2$ must be within the range $[0, \ldots, p-1]$, where $p - m_1 \geq m_1$ and similarly for m_1. $p = 128$ does not satisfy this requirement but $p = 512, 1024$ do.

Therefore there are two possible solutions (only one correct one was needed to get full credit):

- $p = 512$, $f_s =$ 10k-samples, X_m nonzero for $m = 44, 88, 468, 424$
- $p = 1024$, $f_s =$ 20k-samples, X_m nonzero for $m = 44, 88, 980, 936$

6. (6 points) Consider the DT system shown in the figure below.

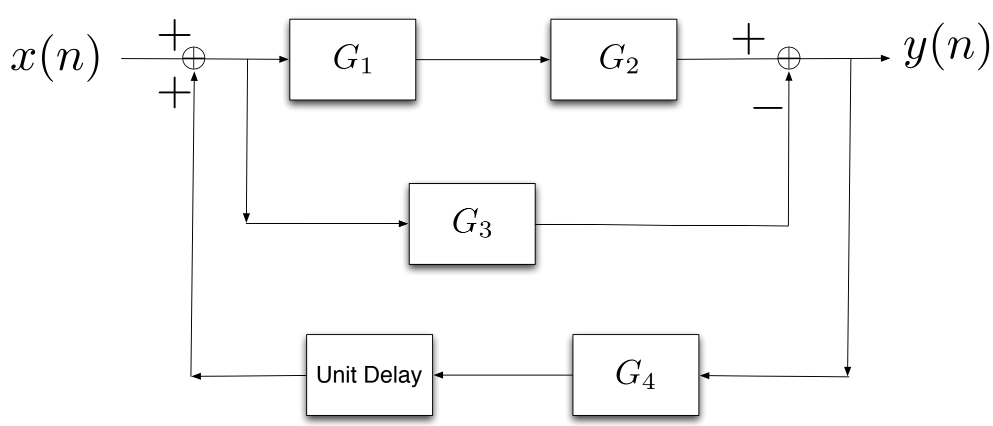


Figure 2: Composition of Systems

a) (4 points) Compute the frequency response H of the overall system in terms of the frequency responses of the subsystems.

b) (2 points) If all the subsystems are causal, must the overall system be causal? (Respond in "yes", "no" or "don't know", no explanation required. A correct answer gets 2 points, incorrect answer gets 0 point, and "don't know" gets 1 point.)

Solution:

a)

$$H(\omega) = \frac{G_1(\omega)G_2(\omega) - G_3(\omega)}{1 + [G_1(\omega)G_2(\omega) - G_3(\omega)]G_4(\omega)e^{-i\omega}}$$

b) Yes.

7. (8 points) Consider a CT LTI system with the following input output relationship:

$$y(t) = \int_0^\infty e^{-s/T} x(t-s) ds.$$

a) (4 points) Compute the frequency response of this system. Plot its magnitude as a function of frequency.

b) (1 point) Interpreting this system as a filter, what kind of filter is this?

c) (3 points) What should be the unit of the parameter T? Explain qualitatively what happens to the frequency response as T is varied. Give an interpretation of the parameter T.

Solution:

a) Let the input be $x(t) = e^{i\omega t}$ and the output be $y(t) = H(\omega)e^{i\omega t}$. Then,

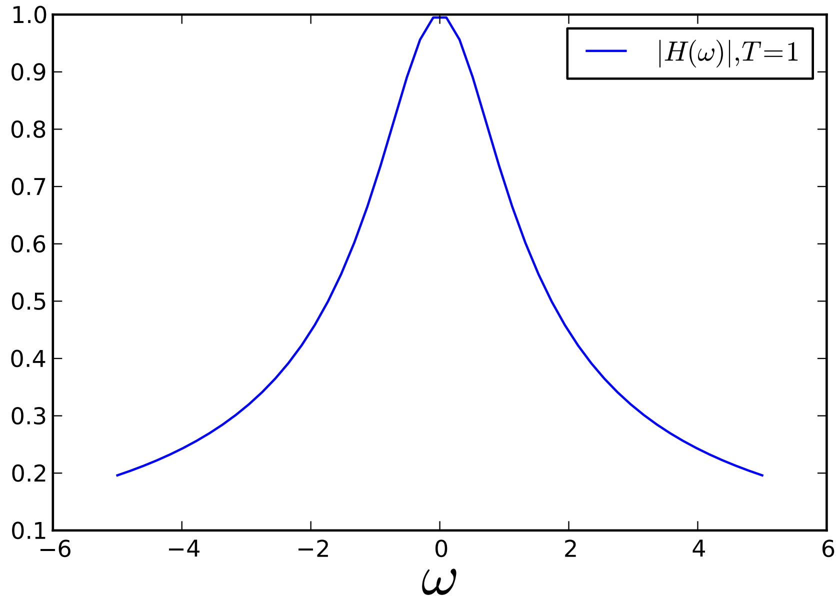
$$\begin{aligned} H(\omega)e^{i\omega t} &= \int_0^\infty e^{-s/T} e^{i\omega(t-s)} ds \\ &= e^{i\omega t} \int_0^\infty e^{-s/T} e^{-i\omega s} ds \\ &= e^{i\omega t} \int_0^\infty e^{-s(1/T+i\omega)} ds \\ &= e^{i\omega t} \left[-\frac{e^{-s(\frac{1}{T}+i\omega)}}{\frac{1}{T}+i\omega} \right]_0^\infty \\ &= \frac{e^{i\omega t}}{\frac{1}{T}+i\omega} \end{aligned}$$

Thus, $H(\omega) = \frac{1}{\frac{1}{T}+i\omega}$ and $|H(\omega)| = \frac{1}{\sqrt{\frac{1}{T^2}+\omega^2}}$.

b) This is a low-pass filter.

c) T should have unit of time. If we set $\omega_0 = \frac{1}{T}$, we have that $\frac{|H(\omega_0)|}{|H(0)|} = \frac{1}{\sqrt{2}}$. Thus, as T is increased, ω_0 falls and so the filter becomes increasingly low pass. Also, the magnitude of the frequency response at $\omega = 0$ also grows with T. We can interpret T as a fuzzy measure of the past window over which the input is integrated to obtain the output.

8. (5 points) Consider a causal LTI system. The output of the system given a periodic input $x(n)$ with period p is an output $y(n)$. Define:



$$\tilde{x}(n) = \begin{cases} x(n) & n = 0, 1, \ldots, p-1 \\ 0 & \text{else} \end{cases}$$
$$\tilde{y}(n) = \begin{cases} y(n) & n = 0, 1, \ldots, p-1 \\ 0 & \text{else} \end{cases}$$

Is $\tilde{y}(n)$ the output of the (same) LTI system when the input is $\tilde{x}(n)$? If "yes", give a proof. If "no", give a counter-example.

Solution:

The output of the system with input $\tilde{x}(n)$ is not $\tilde{y}(n)$ in general. Note that if the impulse response of the system $h(n)$ has length more than 1, then under input $x(n)$, there will be a "spill from the past" coming in at $n = 0$. However, if the input is $\tilde{x}(n)$, there will be no spill coming in from the past at $n = 0$, because $\tilde{x}(n)$ is 0 for all time $n < 0$.

As a counterexample, consider the $x(n)$ given by

$$x(n) = \begin{cases} 6 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd,} \end{cases}$$

and a system with impulse response $h(n) = \delta(n) + \frac{1}{2}\delta(n-1)$. Here, $x(n)$ is periodic with period 2.

Then,

$$y(n) = \begin{cases} 7 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd,} \end{cases}.$$

Then,

$$\tilde{x}(n) = \begin{cases} 6 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 0 & \text{else} \end{cases},$$

$$\tilde{y}(n) = \begin{cases} 7 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 0 & \text{else} \end{cases}.$$

However, the output of the system under input $\tilde{x}(n)$ is given by

$$y_{\mathsf{output}}(n) = \begin{cases} 6 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{else} \end{cases},$$

which is not the same as $\tilde{y}(n)$.