Solution Midterm 2, Math 53, Summer 2012

1. **(a)** (10 points) Let $f(x, y, z)$ be a differentiable function of three variables and define

$$F(s, t) = f(st^2, s + t, s^2 - t).$$

Calculate the partial derivatives F_s and F_t in terms of the partial derivatives of f.

(b) (10 points) Compute the tangent plane to the surface $z = \sqrt{x^3 + y^2}$ at the point $(4, 6, 10)$.

Solution:

(a) Using the chain rule

$$F_s = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} + f_z \frac{\partial z}{\partial s} = t^2 f_x + f_y + 2s f_z,$$

and

$$F_t = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} + f_z \frac{\partial z}{\partial t} = 2st f_x + f_y - f_z.$$

Note that each f_x, f_y, f_z is evaluated at $(x, y, z) = (st^2, s + t, s^2 - t)$.

(b) Letting $f(x, y) = \sqrt{x^3 + y^2}$ the tangent plane has equation

$$z = f(4, 6) + f_x(4, 6)(x - 4) + f_y(4, 6)(y - 6).$$

Now

$$f_x(x, y) = \frac{3x^2}{2\sqrt{x^3 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^3 + y^2}}$$

so

$$f(4, 6) = 10, \quad f_x(4, 6) = \frac{12}{5}, \quad f_y(4, 6) = \frac{3}{5}.$$

Then the equation of the tangent plane is

$$z = 10 + \frac{12}{5}(x - 4) + \frac{3}{5}(y - 6)$$

or equivalently

$$12x + 3y - 5z = 16.$$

2. (20 points) Let $f(x,y) = 2y^3 + x^2y + x^2 + 5y^2$.

   (a) (10 points) Find all critical points of f.

   (b) (10 points) Classify the critical points as local maximum, local minimum or saddle point using the second derivatives test.

(a) Setting the partial derivatives equal to zero gives

$$f_x = 2xy + 2x = 0 \Leftrightarrow x(y+1) = 0 \quad (1)$$
$$f_y = 6y^2 + x^2 + 10y = 0 \quad (2)$$

From (1) we obtain $x = 0$ or $y = -1$.

If $x = 0$: From (2) $y(3y+5) = 0$ so $y = 0$ or $y = -\frac{5}{3}$. We obtain the critical points $(0,0)$ and $(0, -\frac{5}{3})$.

If $y = -1$: From (2) $x^2 = 4$ so $x = \pm 2$ and we get the critical points $(2,-1)$ and $(-2,-1)$.

Critical points: $(0,0)$, $(0,-\frac{5}{3})$, $(2,-1)$, $(-2,-1)$.

(b) The second order partial derivatives are

$$f_{xx} = 2y + 2,\ f_{yy} = 12y + 10,\ f_{xy} = f_{yx} = 2x.$$

Then $D(x,y) = (2y+2)(12y+10) - 4x^2$. Evaluating at the critical points

- $D(0,0) = 20 > 0$, $f_{xx}(0,0) = 2 > 0$. Then $(0,0)$ is a local minimum.
- $D(0,-\frac{5}{3}) = \frac{40}{3} > 0$, $f_{xx}(0,-\frac{5}{3}) = -\frac{4}{3} < 0$. Then $(0,-\frac{5}{3})$ is a local maximum.
- $D(2,-1) = -16 < 0$. Then $(2,-1)$ is a saddle point.
- $D(-2,-1) = -16$. Then $(-2,-1)$ is a saddle point.

3. **(a)** (10 points) Let $a \geqslant 1$ be a constant. Evaluate the integral of the function $f(x, y) = \ln(a^2 + x^2 + y^2)$ over the region D in the plane described by

$$D = \{(x, y) \mid x^2 + y^2 \leqslant 1,\ y \geqslant |x|\}.$$

Hint: It may (or may not) be useful to know that $\int \ln x dx = x \ln x - x + C$.

(b) (10 points) Calculate $\iiint_E z e^{x^2+y^2+z^2} dV$ where E is the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$.

(a) The integral is

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^1 r \ln(a^2 + r^2) dr d\theta = \frac{\pi}{2} \int_0^1 r \ln(a^2 + r^2) dr.$$

Substitute $s = a^2 + r^2$, $ds = 2rdr$ to obtain

$$\frac{\pi}{2} \int_0^1 r \ln(a^2 + r^2) dr = \frac{\pi}{4} \int_{a^2}^{1+a^2} \ln s\, ds.$$

Using the hint, the value of the integral is

$$\frac{\pi}{4}((1 + a^2) \ln(1 + a^2) - a^2 \ln(a^2) - 1).$$

(b) Using cylindrical coordinates, the cone becomes $z = r$ and the integral is

$$\begin{aligned}
\iiint_E z e^{x^2+y^2+z^2} dV &= \int_0^{2\pi} \int_0^1 \int_0^z z e^{r^2+z^2} r dr dz d\theta = \int_0^{2\pi} \int_0^1 z e^{z^2} \frac{e^{r^2}}{2} \Big|_{r=0}^{r=z} dz d\theta \\
&= \pi \int_0^1 z e^{2z^2} - z e^{z^2} dz \\
&= \pi \left(\frac{e^{2z^2}}{4} - \frac{e^{z^2}}{2} \right) \Big|_0^1 \\
&= \frac{\pi}{4}(e^2 - 2e + 1) \\
&= \frac{\pi}{4}(e - 1)^2.
\end{aligned}$$

4. (20 points) Let R be the region in the plane bounded by the lines $y = 1 - x$, $y = 2 - x$ and the hyperbola $xy = \frac{1}{16}$. Calculate

$$\iint_R 2y\,dA,$$

using the change of variables $u = x + y$, $v = x - y$.

The inverse of the transformation is $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

We calculate the image of R in the uv-plane by mapping its boundary. The lines $x + y = 1$ and $x + y = 2$ map to the lines $u = 1$ and $u = 2$ respectively. For the hyperbola

$$\frac{1}{16} = xy = \frac{u+v}{2}\frac{u-v}{2} = \frac{u^2 - v^2}{4},$$

then $u^2 - v^2 = \frac{1}{4}$ which is a hyperbola. Soving for v gives $v = \pm\sqrt{u^2 - \frac{1}{4}}$.

The function $2y$ equals $u - v$. With this the integral is

$$\begin{aligned}
\iint_R 2y\,dA &= \int_1^2 \int_{-\sqrt{u^2-\frac{1}{4}}}^{\sqrt{u^2-\frac{1}{4}}} (u-v)\frac{1}{2}dvdu = \int_1^2 u\sqrt{u^2 - \frac{1}{4}}du \\
&= \frac{1}{3}\left(u^2 - \frac{1}{4}\right)^{3/2}\Big|_1^2 \\
&= \frac{1}{8}(5\sqrt{15} - \sqrt{3}) \\
&= \frac{\sqrt{3}}{8}(5\sqrt{5} - 1).
\end{aligned}$$

5. (20 points) Let I denote the integral $I = \int_0^1 \int_0^z \int_x^z ze^{-y^2}\,dydxdz$.

(a) (10 points) Rewrite the integral in the following orders $dydzdx$, $dzdydx$ and $dxdydz$.

(b) (10 points) Evaluate I.

(a) For the order $dydzdx$ switch the last two variables in the expression for I. This gives

$$I = \int_0^1 \int_x^1 \int_x^z ze^{-y^2}\,dydzdx.$$

For the order $dzdydx$ switch the y and z in the previous expression for I taking x as a constant

$$I = \int_0^1 \int_x^1 \int_y^1 ze^{-y^2}\,dzdydx.$$

For the order $dxdydz$ we can go back to the original expression of I and switch x and y,

$$I = \int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dxdydz.$$

(b) Using the last expression from (a)

$$\begin{aligned} I &= \int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dxdydz = \int_0^1 \int_0^z yze^{-y^2}\,dydz = \int_0^1 -\frac{z}{2}e^{-y^2}\Big|_0^z dydz \\ &= \frac{1}{2}\int_0^1 -ze^{-z^2} + zdz = \frac{1}{2}\Big(\frac{z^2}{2} + \frac{e^{-z^2}}{2}\Big)\Big|_0^1 = \frac{e^{-1}}{4} \\ &= \frac{1}{4e}. \end{aligned}$$

6. (20 points) Let $f(x,y,z) = x^2 + y^2 + z^2$. Find all solutions (x,y,z) of the system of equations coming from minimizing $f(x,y,z)$ subject to the constraint $yz - \dfrac{x^3}{3} = 1$ using the method of Lagrange multipliers.

Then find the point (or points) where the minimum happens and write what that minimum value is.

Let $g(x,y,z) = yz - \frac{x^3}{3}$, so that the restriction is $g(x,y,z) = 1$. Then $\nabla f = \langle 2x, 2y, 2z\rangle$ and $\nabla g = \langle -x^2, z, y\rangle$. The system of eqautions for the method of Lagrange multipliers is $\nabla f = \lambda \nabla g$,

$$\begin{aligned} 2x &= -\lambda x^2 && (1) \\ 2y &= \lambda z && (2) \\ 2z &= \lambda y && (3) \\ yz - \frac{x^3}{3} &= 1 && (4) \end{aligned}$$

Using (2) and (3) we obtain $4y = \lambda^2 y$, that is $(4 - \lambda^2)y = 0$ from where $y = 0$ or $\lambda = 2$ or $\lambda = -2$. We study each case.

Case 1: $y = 0$, then from (3), $z = 0$. From (4), $x^3 = -3$, so $x = -3^{1/3}$. We obtain the point

$$(-3^{1/3}, 0, 0).$$

Case 2: $\lambda = 2$, then from (1), $x = -x^2$ and from (2), $y = z$. For $x = -x^2$ we have that either $x = 0$ or $x = -1$.

Subcase 1: $x = 0$ and $y = z$. From (4), $y^2 = 1$, so $y = \pm 1 = z$ and we obtain the points

$$(0, 1, 1),\ (0, -1, -1).$$

Subcase 2: $x = -1$ and $y = z$. From (4), $y^2 = \frac{2}{3}$, so $y = \pm\sqrt{\frac{2}{3}} = z$ and we obtain the points

$$(-1, \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}),\ (-1, -\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}).$$

Case 3: $\lambda = -2$. Then from (1), $x = x^2$ and from (2), $y = -z$. For $x = x^2$ we have that either $x = 0$ or $x = 1$. In either case, from (4) we obtain $y^2 = -1$ for the case $x = 0$ and $y^2 = -\frac{4}{3}$ for the case $x = 1$, none of which has a solution.

There are five solution to the system of equations:

$$(-3^{1/3}, 0, 0),\ (0, 1, 1),\ (0, -1, -1),\ (-1, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}),\ (-1, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}).$$

Evaluating the function

$$f(0,1,1) = 2,\ f(0,-1,-1) = 2,\ f(-1, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}) = \frac{7}{3},\ f(-1, -\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}) = \frac{7}{3},$$
$$f(-3^{1/3}, 0, 0) = 3^{2/3}.$$

The minimum is attained at $(0,1,1)$ and $(0,-1,-1)$ and the minimum value is 2.

7. (20 points) If you take the circle $(y-\frac{1}{2})^2+z^2=\frac{1}{4}$ in the yz-plane and rotate it about the z-axis, the resulting surface is called torus. Its equation in spherical coordinates is $\rho=\sin\phi$. The surface of equation $\rho=\cos\phi$ is a sphere.

(a) (4 points) Convert the equation of the sphere $\rho=\cos\phi$ to cartesian coordinates and identify its radius and center.

(b) (16 points) Calculate the mass of the solid E that is inside the sphere $\rho=\cos\phi$ and outside the torus $\rho=\sin\phi$ if the density equals $\sigma(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$.

(a) Using that $\rho\cos\phi=z$ we get $\rho=\frac{z}{\rho}$ that is $\rho^2=z$. Then $x^2+y^2+z^2=z$. Completing square gives

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4},$$

a sphere of radius $\frac{1}{2}$ centered at $(0,0,\frac{1}{2})$.

(b) We see that the angle θ moves from 0 to 2π. To find the range of ϕ we find the intersection of $\rho=\sin\phi$ and $\rho=\cos\phi$, that is we set $\sin\phi=\cos\phi$ giving $\phi=\frac{\pi}{4}$. The description of E in spherical coordinates is

$$E=\{(\rho,\phi,\theta)|0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\phi\leqslant\frac{\pi}{4},\ \sin\phi\leqslant\rho\leqslant\cos\phi\}.$$

The density in spherical coordinates is $\sigma(\rho,\phi,\theta)=\frac{1}{\rho}$. The total mass m is

$$\begin{aligned}
m&=\iiint_E\sigma dV=\int_0^{\frac{\pi}{4}}\int_{\sin\phi}^{\cos\phi}\int_0^{2\pi}\frac{1}{\rho}\rho^2\sin\phi d\theta d\rho d\phi\\
&=2\pi\int_0^{\frac{\pi}{4}}\int_{\sin\phi}^{\cos\phi}\rho\sin\phi d\theta d\rho d\phi=\pi\int_0^{\frac{\pi}{4}}\rho^2\Big|_{\sin\phi}^{\cos\phi}\sin\phi d\phi\\
&=\pi\int_0^{\frac{\pi}{4}}(\cos^2\phi-\sin^2\phi)\sin\phi d\phi=\pi\int_0^{\frac{\pi}{4}}(2\cos^2\phi-1)\sin\phi d\phi\\
&=\pi\left(-\frac{2}{3}\cos^3\phi+\cos\phi\right)\Big|_0^{\frac{\pi}{4}}\\
&=\frac{\pi}{3}(\sqrt{2}-1).
\end{aligned}$$