Math 53, Multivariable Calculus, Summer 2012

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July 9, 2012

MIDTERM 1

8:10-9:50

Name: Solution

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Directions: Please do not start until instructed to, and do not work after time has been canceled. Calculators, smartphones and "cheat sheets" are not allowed. Please show your work for every problem and justify your steps. Partial credit may be given if the work merits it. Write your answer inside a $\boxed{box}$.

Cross out any work that you do not want to be graded. Scratch paper will be provided, you can ask for more if you need.

The last page contains some trigonometric identities and values of sine and cosine. You can remove the page if you want and use it as scratch paper.

Each question is worth 20 points

Please do not mark the table below!

Question 1	20
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Question 7	20
Total	140

1. Let $f(x,y) = \frac{x+y^2}{x-y^2}$

(a) Find the domain of the function. The level curves $f(x,y) = c$, for $c \neq 1$ are a kind of conic section, identify them. Identify the level curve for $c = 1$.

(b) Calculate the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Simplify the expressions.

(a) Domain: all (x,y) such that $x \neq y^2$

$= \{(x,y) : x \neq y^2\}$ or equivalently $\{(x,y) : x > y^2$ or $x < y^2\}$

Level curves: $f(x,y) = c \Leftrightarrow \dfrac{x+y^2}{x-y^2} = c \Leftrightarrow x + y^2 = c(x - y^2)$

$\Leftrightarrow (1-c)x = -(1+c)y^2 \quad (*)$

If $c \neq 1$ then $x = -\dfrac{(1+c)}{1-c} y^2 \Big\}$ A family of parabolas (and $(x,y) \neq (0,0)$)

If $c = 1$ then from $(*)$: $0 = -2y^2 \Leftrightarrow y = 0 \leftarrow$ x-axis (with the point $(0,0)$ removed as it's not in the domain).

(b)

$$\frac{\partial f}{\partial x} = \frac{x - y^2 - (x+y^2)}{(x-y^2)^2} = \frac{-2y^2}{(x-y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y(x-y^2) - (x+y^2)(-2y)}{(x-y^2)^2} = \frac{4xy}{(x-y^2)^2}$$

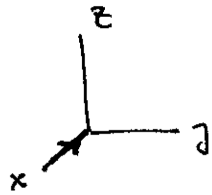
2. (a) Find the limit, if it exists, or show that the limit does not exist

$$\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^4}{x^4+y^4+z^4}$$

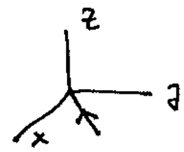
(b) Let $f(x,y) = \begin{cases} \frac{x^2\sin^2(y)}{3x^2+2y^2} & \text{if } (x,y)\neq(0,0) \\ \alpha & \text{if } (x,y)=(0,0) \end{cases}$

Is there a real number α such that f is continuous at $(0,0)$?

(a) Through the x-axis : $y=z=0$



$$\lim_{x\to 0} \frac{x^4}{x^4} = 1$$



Through the diagonal $x=y,\ z=0$

$$\lim_{x\to 0} \frac{(2x)^4}{2x^4} = \lim_{x\to 0} \frac{16x^4}{2x^4} = 8$$

Since $8\neq 1$ we have different limits for two paths through $(0,0,0)$ $\Rightarrow$ The limit does not exist.

(b) We need to calculate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{3x^2+2y^2}$$

Using polar coordinates : $= \lim\limits_{r\to 0^+} \dfrac{\cancel{r^2}\cos^2\theta\ \sin^2(r\sin\theta)}{3\cancel{r^2}\cos^2\theta + 2\cancel{r^2}\sin^2\theta}$

$$= \lim_{r\to 0^+} \frac{\cos^2\theta}{3\cos^2\theta+2\sin^2\theta}\sin^2(r\sin\theta) = \lim_{r\to 0^+} \underbrace{\frac{\cos^2\theta}{2+\cos^2\theta}}_{\text{bounded between } 0 \text{ and } \frac{1}{2}} \underbrace{\sin^2(r\sin\theta)}_{\text{goes to zero as } r\to 0^+}$$

$= 0$ as $\sin^2(r\sin\theta) \xrightarrow[r\to 0^+]{} 0$ and $\dfrac{\cos^2\theta}{2+\cos^2\theta}$ is bounded.

Therefore if $\boxed{\alpha=0}$ we get $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ and f is continuous

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(b) Can also use the squeeze theorem:

$$0 \le \frac{x^2 \sin^2(y)}{3x^2+2y^2} = \underbrace{\frac{x^2}{3x^2+2y^2}}_{\le \frac{1}{3}} \sin^2(y) \le \frac{x^2}{3x^2} \sin^2(y) = \frac{1}{3}\sin^2(y)$$

since $\lim\limits_{(x,y)\to(0,0)} \frac{1}{3}\sin^2(y) = \sin^2(0) = 0$

we get $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 \sin^2(y)}{3x^2+2y^2} = 0.$

3. Consider the polar curves $r = 1 + \sin(3\theta)$ and $r = 1$. Let D denote the region in the first quadrant inside the first curve and outside the second curve.

   (a) The cartesian graph of the function $r = 1 + \sin(3\theta)$, for $0 \leqslant \theta \leqslant 2\pi$, is given. Use it to make a **good** sketch the curve in the graph provided. A **good** graph includes orientation and the angles where $r = 0$ are marked by doted lines.

   (b) Find the area of D (if you sketch D do not use the graph in (a), make a new one).

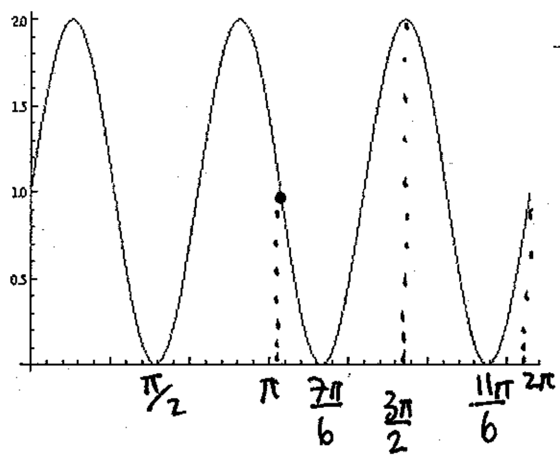
   (c) Write but do not evaluate the integral that computes the perimeter of D.

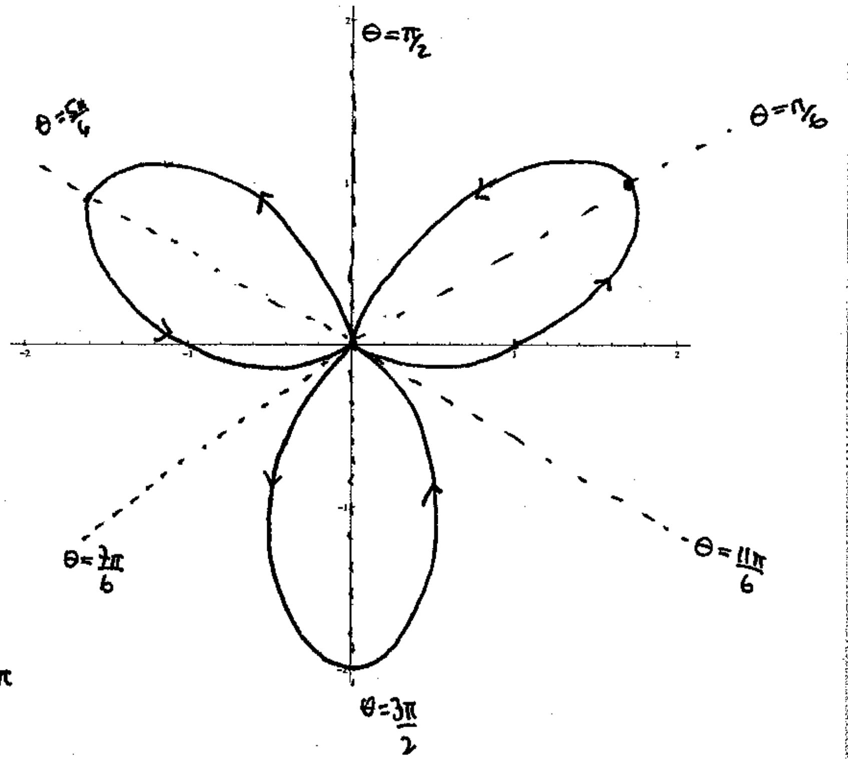
(a) $r=0 \iff 1+\sin(3\theta)=0$

$\iff \sin(3\theta) = -1$

$\iff 3\theta = \frac{3\pi}{2}, \frac{3\pi}{2}+2\pi, \frac{3\pi}{2}+4\pi, \dots$

$\iff \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$



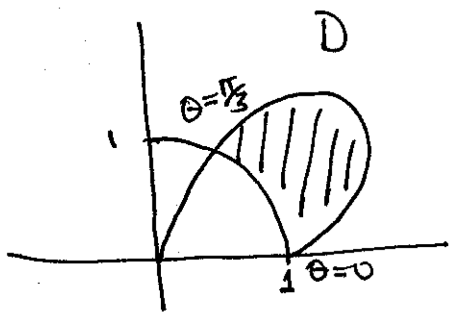


$r(\pi) = 1, \quad r\left(\frac{3\pi}{2}\right) = 2$

$r(\theta) = 2 \iff \sin(3\theta)=1 \iff 3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, \dots$

(b)



Intersection $1+\sin(3\theta) = 1$

$\iff \sin(3\theta) = 0$

$\iff 3\theta = 0, \pi, \dots$

$\underbrace{\theta = 0, \tfrac{\pi}{3}}_{\text{in 1st quadrant}}, \dots$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} (1+\sin(3\theta))^2\,d\theta - \frac{1}{2}\int_0^{\pi/3} 1^2\,d\theta = \frac{1}{2}\int_0^{\pi/3} \cancel{1}+2\sin(3\theta)+\underbrace{\sin^2(3\theta)}_{\frac{1-\cos(6\theta)}{2}} - \cancel{1}\,d\theta$$

$$= \frac{1}{2}\left[-\frac{2}{3}\cos(3\theta) + \frac{1}{2}\theta - \frac{\sin(6\theta)}{12}\right]_0^{\pi/3} = \frac{1}{2}\left[\left(\frac{2}{3}+\frac{\pi}{6}\right) - \left(-\frac{2}{3}\right)\right] = \boxed{\frac{1}{2}\left(\frac{\pi}{6}+\frac{4}{3}\right)}$$

3.(b)

$$\boxed{\text{Area} = \frac{1}{2}\left(\frac{\pi}{6} + \frac{4}{3}\right)}$$

(c) Perimeter $= \int_0^{\pi/3} \sqrt{(1+\sin(3\theta))^2 + (3\cos(3\theta))^2}\, d\theta + \int_0^{\pi/3} \sqrt{1^2+0^2}\, d\theta$

$$= \int_0^{\pi/3} \sqrt{1 + 2\sin(3\theta) + \sin^2(3\theta) + 9\cos^2(3\theta)}\, d\theta + \frac{\pi}{3}.$$

44. Let $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$.

(a) Calculate the arclength function starting from $t = 0$.

(b) Reparametrize $\mathbf{r}(t)$ with respect to arclength.

(c) A particle follows the trajectory $\mathbf{r}(t)$ starting from the point $(1, 0, 1)$ until it has traveled $\sqrt{3}$ units of distance. It then falls vertically to the xy-plane. Find the point in the xy-plane where it falls to.

(a) $r' = \langle e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t \rangle$

$$|r'| = \sqrt{\underbrace{e^{2t}\cos^2 t + e^{2t}\sin^2 t}\ - \cancel{2e^{2t}\sin t\cos t} + e^{2t}\sin^2 t + e^{2t}\cos^2 t + \cancel{2e^{2t}\sin t\cos t} + e^{2t}}$$

$$= \sqrt{e^{2t} + e^{2t} + e^{2t}} = \sqrt{3}\, e^t$$

$$s(t) = \int_0^t \sqrt{3}\, e^t\, dt = \sqrt{3}(e^t - 1) \qquad \boxed{s(t) = \sqrt{3}\,(e^t - 1)}$$

(b) $$e^t = \frac{s}{\sqrt{3}} + 1 \qquad t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$

$$\vec{r}(s) = \left\langle \left(\frac{s}{\sqrt{3}}+1\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right),\ \left(\frac{s}{\sqrt{3}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right),\ \frac{s}{\sqrt{3}}+1 \right\rangle$$

(c) $(1,0,1) \leftrightarrow t = 0$

Need $\vec{r}$ of $s = \sqrt{3}$

$$\vec{r}(s=\sqrt{3}) = \langle 2\cos(\ln 2),\ 2\sin(\ln 2),\ 2 \rangle$$

$\Rightarrow$ Point in the xy-plane is $\boxed{(2\cos(\ln 2), 2\sin(\ln 2), 0)}$.

5. (a) Parametrize the curve of intersection of the surfaces $y = x^2 + z$ and $x^2 + z^2 = 1$. Indicate the domain of the parameter you are using.

(b) Find parametric equations for the tangent line to the curve in (a) at the point $(1, 1, 0)$.

(a) The x & z variables move in a circle of radius 1

→ parametrize as

$$x = \cos t \qquad 0 \le t \le 2\pi$$
$$z = \sin t$$

$$y = \cos^2 t + \sin t$$

$$\Rightarrow \begin{cases} x = \cos t \\ y = \cos^2 t + \sin t \\ z = \sin t \end{cases}, \quad 0 \le t \le 2\pi$$

$$r(t) = \langle \cos t, \cos^2 t + \sin t, \sin t \rangle$$

(b) $\vec{r}\,'(t) = \langle -\sin t, -2\cos t \sin t + \cos t, \cos t \rangle$

The point $(1,1,0)$ corresponds to $t = 0$ (or 2π is the same)

$\Rightarrow \vec{r}\,'(0) = \langle 0, 1, 1 \rangle$

$\Rightarrow$ tangent line

$$\begin{cases} x = 1 \\ y = 1 + t \\ z = t \end{cases}$$

6. (a) Find parametric equations for the line through the point $(2,0,2)$, parallel to the plane $x+y+z=10$ and orthogonal to the line of parametric equations $x=1+t,\ y=1-t,\ z=2t$.

(b) Let θ denote the angle between the planes $2x+2y+z=1$ and $-x+2y-2z=2$. The planes intersect in a line that we call L. Find the plane that contains L and forms acute angle $\theta/2$ with both planes.

(a) Point $(2,0,2)$

Direction: v

Since the line is parallel to $x+y+z=10$ of normal vector $n = \langle 1,1,1\rangle$ then

v & n are orthogonal.

The line is orthogonal to the given line of direction

$u = \langle 1,-1,2\rangle$

$\Rightarrow$ v & u are orthogonal

Can obtain v as $v = n \times u = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3,-1,-2\rangle$

$\Rightarrow$ Parametric equations:

$$\begin{cases} x = 2+3t \\ y = -t \\ z = 2-2t \end{cases}$$

(b) Normal to first plane: $n_1 = \langle 2,2,1\rangle$

" " second " : $n_2 = \langle -1,2,-2\rangle$

We can calculate the line (though it's not needed)

Direction $v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & 2 & 1 \\ -1 & 2 & -2 \end{vmatrix} = \langle -6, 3, 6\rangle$ or we can use $\langle -2,1,2\rangle$

Point:

set $z=0$ to get

$$\begin{aligned} 2x+2y&=1 \\ -x+2y&=2 \end{aligned} \quad \Rightarrow \quad 6y=5$$

$$y=\frac{5}{6}, \quad x=2y-2=\frac{5}{3}-2 = -\frac{1}{3}$$

$\Rightarrow$ Point $\left(-\frac{1}{3}, \frac{5}{6}, 0\right)$

line : $\langle -\frac{1}{3}, \frac{5}{6}, 0\rangle + t\langle -2,1,2\rangle$

<u>Angle θ</u> :

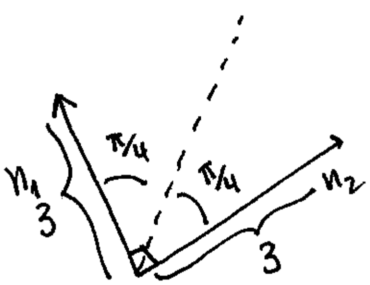
$$\cos(\theta) = \frac{n_1 \cdot n_2}{|n_1|\,|n_2|} = \frac{-2+4-2}{|n_1|\,|n_2|} = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

$$|n_1| = |n_2| = 3$$

We need a normal vector n: It should form $\frac{\pi}{4}$ angle with n_1 & n_2 and contain L

For it to contain L we need n orthogonal to $v = \langle -2,1,2\rangle$



We see that the diagonal of the parallelogram formed by n_1 & n_2 forms angle $\frac{\pi}{4}$ with both vectors since they have the same magnitude.

diagonal $= n_1 + n_2 = \langle 1,4,-1\rangle$

We claim that the plane with $n=\langle 1,4,-1\rangle$ and passing through $\left(-\frac{1}{3}, \frac{5}{6}, 0\right)$ works:

$$x+4y-z = -\frac{1}{3} + \cancel{4}^{2}\cdot\frac{5}{\cancel{6}_{3}} + 0 = 3, \quad \boxed{x+4y-z=3}$$

Check: put the coordinates of the lines:

$\left(-\frac{1}{3}-2t\right) + 4\left(\frac{5}{6}+t\right) - (2t) = 3$ ✓ & Forms angle $\theta/2$ with the two planes by the way we constructed it.

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(b) Note that in this case, because the angle is $\frac{\pi}{2}$ there is another plane: for this use n_1 & $-n_2$ that also form $\frac{\pi}{2}$ angle

Diagonal is $n_1 - n_2 = \langle 3, 0, 3 \rangle$

$\Rightarrow$ 2^{nd} plane: $3x + 3z = 3 \cdot \left(-\frac{1}{3}\right) + 3 \cdot 0 = -1$

$$\boxed{3x + 3z = -1}$$

If the angle θ is $\theta < \pi/2$ there is only one plane forming acute angle $\frac{\theta}{2}$ with both planes.

7. A particle moves in space following the trajectory $\mathbf{r}(t)$. Suppose that $\mathbf{r}'' = |\mathbf{r}|^2\mathbf{r}$ for all t, $\mathbf{r}(0) = \langle 1,1,1\rangle$ and $\mathbf{r}'(0) = \langle 1,2,4\rangle$. Define $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t)$. As always, in the next questions justify the relevant steps.

(a) Show that $\mathbf{L}$ is constant and find its value.

(b) Calculate $\mathbf{r}\cdot\mathbf{L}$.

(c) The curve $\mathbf{r}(t)$ lies in a plane through the origin. Find the explicit equation of the plane.

(a) We compute $\frac{dL}{dt}$

$$\frac{dL}{dt} = \frac{d}{dt}(r \times r') \underset{\substack{\uparrow \\ \text{use product rule}}}{=} \underbrace{r' \times r'}_{\substack{= 0 \\ \text{because if } a \,\&\, b \\ \text{are parallel then} \\ a \times b = 0}} + r \times r'' = r \times r'' = r \times (|r|^2 r)$$

$= r \times (|r|^2 r)$

$= 0$ because r & $|r|^2 r$ are parallel

Then L is constant and therefore equals to its value at $t=0$

$$L(t) = \langle 1,1,1\rangle \times \langle 1,2,4\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \langle 2,-3,1\rangle$$

$$\boxed{L(t) = \langle 2,-3,1\rangle}$$

(b) $$r \cdot L = r \cdot \underbrace{(r \times r')}_{\substack{\text{orthogonal} \\ \text{to } r \,\&\, r' \\ \Rightarrow \text{ doted with} \\ r \text{ gives zero}}} = 0.$$

(c) From (b) $r \cdot L = 0$ & $L = \langle 2,-3,1\rangle$. If r has components $\langle x,y,z\rangle$, they satisfy $2x - 3y + z = 0$

Plane: $\boxed{2x - 3y + z = 0}$