

FINAL EXAM

No collaboration permitted.

Three sheets of notes, both sides, permitted. Turn them in with your exam.
No access to the internet, textbook, notes beyond the three sheets above is permitted.

You have **till 6PM** to complete the exam.

Be clear and precise in your answers. Illegible exams cannot be graded.

Write your name and student ID on **EVERY** sheet.

There are a total of 179 points on this exam. 120 is a good score.

Goodluck!

Note: Feel free to leave answers to any questions in terms of $\Phi(x)$ or $\Phi^{-1}(x)$.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

1. (24 points) True or False? Prove or give a counterexample.

a. (12 points) If U and V are iid Uniform $[0, 1]$ random variables, and $X = \max(U, V)$, then (U, X) are jointly continuous random variables.

- b. (12 points) If X_i are iid Exponential random variables with parameter λ and N is an independent Geometric random variable with parameter $0 < p < 1$, then $Y = \sum_{i=1}^N X_i$ is an Exponential random variable.

2. (20 points) Consider the interval $[0, 1]$. An arrival process is constructed for this interval as follows:

- First, a random variable N is drawn with PMF $Pr(N = i) = e^{-1} \frac{1}{i!}$ (for integer $i \geq 0$, the PMF is zero for all other values.)
- Then, N arrival times X_i are chosen iid Uniform $[0, 1]$

Let T be the time till the first arrival. (If $N = 0$, then $T = 1$.)

What is the distribution for T ?

(HINT: Think about Poisson Processes to get an intuition. Then try and calculate $P(T > t)$ using conditioning as appropriate.)

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3. (20 points) This problem asks you to construct Markov Chains having certain properties. No justification is required.

a. (10 points) A discrete-time Markov Chain with a unique stationary distribution of $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.
(*HINT: you can solve both a and b with a single solution if you want.*)

b. (10 points) A discrete-time Markov Chain with an eigenvalue of 0 associated with its probability transition matrix.

4. (20 points) Consider X_i iid discrete random variables with a uniform probability over $\{0, 1, 2\}$.
Prove

$$\lim_{n \rightarrow \infty} \Pr(\text{EmpiricalMedian}(X_1, \dots, X_n) = 1) = 1.$$

Recall that the Empirical Median of a sequence is the number in the middle of the sequence after you sort the sequence. Feel free to assume that n is odd.

(HINT: Use the AEP form of the WLLN to see why this must be true. Then prove it however you want.)

5. (40 points) X is a standard Gaussian $N(0, 1)$ random variable. U_i are iid standard Gaussian $N(0, 1)$ random variables that are also independent of X .

a. (5 points) Let $Y_1 = X + U_1$. What is the MMSE estimate for X given Y_1 ? And calculate the variance of the estimation error.

b. (15 points) Suppose that you also have access to $Y_2 = U_1 + U_2$. What is the MMSE estimate for X given both Y_1, Y_2 ? And calculate the variance of the estimation error.

- c. (20 points) Now suppose that $Y_i = U_{i-1} + U_i$ for $i \geq 2$. What is the MMSE estimate for X given Y_1, Y_2, Y_3 ? And calculate the variance of the estimation error. What happens to the optimal estimation error as the number of observations $n \rightarrow \infty$? Why?

(HINT: There are many ways to proceed. Remember the key properties of jointly Gaussian estimation: the MMSE estimate and the estimation error are independent of each other. And if you get a new observation, you don't have to recalculate everything from scratch. . .)

(Second HINT: Problems are easier in the right coordinate system. So as a trick, think about first causally transforming the observations Y_i into a form Y'_i in which the "noise" is independent from observation to observation and the "signal" X is present in each observation. . .)

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6. (30 points) You are attending a party and are immediately introduced to the host and begin a conversation about the election. After some time, they introduce you to someone new (that you haven't met before) and you begin a new conversation. Suppose that there is a social convention that a long-haired person will only introduce you to a random short-haired person and a short-haired person will only introduce you to a random long-haired person. (The host is long-haired.)

Further suppose that a Long-haired person introduces you to someone new in a random amount of time that is distributed like an Exponential random variable with parameter $\lambda = 2$. For Short-haired people, the parameter is $\lambda = 1$.

- a. (10 points) **Argue that the proportion of long-haired people that you've met will be very close to 0.5 of the total people that you've met.**
(*HINT: This is easy.*)

b. (20 points) **About how long a time should you budget for yourself at the party to have met at least 101 people total with a probability of at least 80%?**

(HINT: First calculate the expected amount of time till you meet the 101th person. Then, what do you think the pdf of the time to meet the 101st person looks like?)

(Second HINT: This is part (b) for a reason.)

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7. (25 points) We want to estimate what fraction of people like Psy's Gangnam Style video. The population consists of 50% Long-haired people and 50% Short-haired people.

We are not allowed to poll people by drawing them randomly from the population. Instead, we start with one randomly selected person, and from then on, we randomly pick one of that person's friends. Next we sample that person's friends to pick a person. And so on. . . .

The challenge is that long-haired people tend to have long-haired friends and short-haired people tend to have short-haired friends. Suppose that a long-haired person has $\frac{2}{3}$ long-haired friends and $\frac{1}{3}$ short-haired friends. Similarly, a short-haired person has $\frac{2}{3}$ short-haired friends and $\frac{1}{3}$ long-haired friends.

How many people should we sample using this process to get at least 100 long-haired people with confidence at least 80%?

(HINT: Think about using a Markov chain in your model. What effect do you think this has? Think about the HW where you had to prove a WLLN for Markov chains. . .)

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