Math 121A: Midterm 1

1. The function
$$f(x, y, z) = x^2 + 3y^2 + 5z^2 + 2xy + 4yz - 4x - 2z$$
has one minimum point. Find its location.

2. (a) Calculate the derivative of $f(x) = \log(\log(\log(x)))$.
   (b) By using an appropriate series test, determine whether
   $$\sum_{n=3}^{\infty} \frac{1}{n \log(n) \log(\log(n))}$$
   converges or diverges.
   (c) By using an appropriate series test, determine whether
   $$\sum_{n=3}^{\infty} \frac{(-1)^n}{n \log(n) \log(\log(n))}$$
   converges or diverges.

3. By using Lagrange multipliers, find the smallest possible surface area (including both ends) of a cylinder with volume V.

4. (a) By considering appropriate powers of $e^{i\theta} = \cos\theta + i \sin\theta$ or otherwise, determine an expression for $\sin^3\theta$ as a linear combination of terms with the form $\sin k\theta$.
   (b) Consider the annulus A defined as $a \leq r \leq b$ in polar coordinates, where $0 < a < b$. Show that for any integer k, the function $r^{\pm k} \sin k\theta$ is a solution to the Laplace equation $\nabla^2 \phi = 0$ in A.
   Hint: the Laplacian in polar coordinates is given by
   $$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}.$$
   (c) Find a solution to $\nabla^2 \phi = 0$ in A that satisfies the boundary conditions
   $$\phi(a, \theta) = 4 \sin^3 \theta, \qquad \phi(b, \theta) = 0.$$