Solutions to the Final Exam, Math 53, Summer 2012

1. **(a)** (10 points) Let C be the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 1$ with counterclockwise orientation. Calculate $\int_C (y^2 + e^{\sqrt{x}})dx + xdy$.

(b) (10 points) If the directional derivatives at the point $(1, 1)$ are given

$$D_{\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle} f(1,1) = \sqrt{2}, \quad D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(1,1) = \sqrt{3},$$

find $f_x(1,1)$ and $f_y(1,1)$.

Solution:

(a) Use Green's Theorem. $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = 2y$, so

$$\begin{aligned}
\int_C (y^2 + e^{\sqrt{x}})dx + xdy &= \iint_D 1 - 2y dA = \int_{-1}^{1} \int_{x^2}^{1} 1 - 2y dy dx = \int_{-1}^{1} y - y^2 \Big|_{x^2}^{1} dx \\
&= \int_{-1}^{1} -x^2 + x^4 dx = -\frac{x^3}{3} + \frac{x^5}{5} \Big|_{-1}^{1} = -\frac{2}{3} + \frac{2}{5} \\
&= \boxed{-\frac{4}{15}}.
\end{aligned}$$

(b) The directional derivatives are related to the partial derivatives in the following way $D_{\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle} f = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y$ and $D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f = \frac{1}{2} f_x + \frac{\sqrt{3}}{2} f_y$. Then, evaluating at $(1, 1)$ we obtain the system of equations

$$\begin{aligned}
\frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y &= \sqrt{2} \\
\frac{1}{2} f_x + \frac{\sqrt{3}}{2} f_y &= \sqrt{3},
\end{aligned}$$

where both partial derivatives are evaluated at $(1, 1)$. Solving the system of equations gives

$$\boxed{f_x(1,1) = \sqrt{6} - \sqrt{3},\ f_y(1,1) = 3 - \sqrt{2}}.$$

2. Let S be the surface parametrized by $\mathbf{r}(u,v) = \langle \sin u \cos u, \sin^2 u, v\rangle$ where the domain of the parameters is $D = \{(u,v) | 0 \leqslant u \leqslant \frac{\pi}{2},\ 0 \leqslant v \leqslant \sin^2 u\}$.

(a) (10 points) Find the tangent plane at the point $(x,y,z) = (\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{1}{2})$.

(b) (10 points) Calculate $\displaystyle\iint_S (x+1)dS$.

Solution:

(a) We need to calculate $\mathbf{r}_u \times \mathbf{r}_v$.

$$\mathbf{r}_u = \langle \cos^2 u - \sin^2 u, 2\sin u \cos u, 0\rangle,\ \mathbf{r}_v = \langle 0,0,1\rangle,$$

so $\mathbf{r}_u \times \mathbf{r}_v = \langle 2\sin u \cos u, \sin^2 u - \cos^2 u, 0\rangle$. The point $(x,y,z) = (\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{1}{2})$ corresponds to $u = \frac{\pi}{6}$, $v = \frac{1}{2}$. Then the normal vector to the plane is

$$\mathbf{r}_u \times \mathbf{r}_v(\frac{\pi}{6}, \frac{1}{2}) = \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\rangle.$$

The equation of the tangent plane is $\frac{\sqrt{3}}{2}(x - \frac{\sqrt{3}}{4}) - \frac{1}{2}(y - \frac{1}{4}) = 0$ or simplified

$$\boxed{2\sqrt{3}x - 2y = 1}.$$

(b) $\displaystyle\iint_S (x+1)dS = \iint_D (\sin u \cos u + 1)|\mathbf{r}_u \times \mathbf{r}_v| dudv$. The magnitude of the normal vector is

$$|\mathbf{r}_u \times \mathbf{r}_v| = (4\sin^2 u \cos^2 u + (\sin^2 u - \cos^2 u)^2)^{1/2} = (\sin^4 u + 2\sin^2 u \cos^2 u + \cos^4 u)^{1/2}$$

that simplifies to $|\mathbf{r}_u \times \mathbf{r}_v| = ((\sin^2 u + \cos^2 u)^2)^{1/2} = 1$. Then

$$\begin{aligned}\iint_S (x+1)dS &= \int_0^{\frac{\pi}{2}} \int_0^{\sin^2 u} (\sin u \cos u + 1) dvdu = \int_0^{\frac{\pi}{2}} \sin^3 u \cos u + \sin^2 u \, du \\ &= \int_0^{\frac{\pi}{2}} \sin^3 u \cos u + \frac{1}{2}(1 - \cos(2u))\, du = \frac{\sin^4 u}{4} + \frac{u}{2} - \frac{\sin(2u)}{4}\Big|_0^{\frac{\pi}{2}} \\ &= \boxed{\frac{1+\pi}{4}}.\end{aligned}$$

3. (20 points) Define $\mathbf{G} = \langle 2zxe^{x^2-y^2}, -2zye^{x^2-y^2}, e^{x^2-y^2} + 2z\rangle$, $\mathbf{H} = \langle 0, x, -y\rangle$ and $\mathbf{F} = \mathbf{G} + \mathbf{H}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment from $(1, 2, 4)$ to $(-1, 1, 1)$.

Hint: Calculate the line integrals for **G** and **H** separately. Use a different method for each integral.

Solution:

(a) The vector field **G** is conservative. We look for a potential:

$$f_x = 2zxe^{x^2-y^2} \Rightarrow f = ze^{x^2-y^2} + g(y, z) \Rightarrow f_y = -2zye^{x^2-y^2} + g_y(y, z),$$

Then $g_y = 0$ giving $g(y, z) = h(z)$, so

$$f = ze^{x^2-y^2} + h(z) \Rightarrow f_z = e^{x^2-y^2} + h'(z).$$

Then $h'(z) = 2z$ giving $h = z^2 + c$, where c is a constant. A potential for **G** is $f(x, y, z) = ze^{x^2-y^2} + z^2$. By the fundamental theorem of line integrals

$$\int_C \mathbf{G} \cdot d\mathbf{r} = f(-1, 1, 1) - f(1, 2, 4) = -14 - 4e^{-3}.$$

For **H** we evaluate the integral directly. A parametrization of C is $\mathbf{r}(t) = \langle 1, 2, 4\rangle + t\langle -2, -1, -3\rangle = \langle 1 - 2t, 2 - t, 4 - 3t\rangle$, $0 \leqslant t \leqslant 1$. Then

$$\begin{aligned}\int_C \mathbf{H} \cdot d\mathbf{r} &= \int_0^1 \langle 0, 1 - 2t, -2 + t\rangle \cdot \langle -2, -1, -3\rangle dt = \int_0^1 5 - t\, dt \\ &= 5 - \frac{1}{2}.\end{aligned}$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -9 - \frac{1}{2} - 4e^{-3} = \boxed{-\frac{19}{2} - 4e^{-3}}.$$

4. (20 points) Let S be the ellipsoid of equation $x^2+\frac{y^2}{2}+\frac{z^2}{3}=1$ and let (u,v,w) be a point in S with $u>0$, $v>0$ and $w>0$.

The tangent plane to S at (u,v,w) has equation $ux+\frac{vy}{2}+\frac{wz}{3}=1$ and together with the three coordinate planes encloses a (pyramid-like) solid E whose volume equals $\frac{1}{uvw}$.

Find the point (u,v,w) as in the first paragraph such that E has the minimum possible volume. Write what that volume is.

Solution:

The problem is to minimize $\frac{1}{uvw}$ subject to the constraint $u^2+\frac{v^2}{2}+\frac{w^2}{3}=1$, with $u,v,w>0$. Using Lagrange multipliers,

$$-\frac{1}{u^2vw}=2\lambda u,\ -\frac{1}{uv^2w}=\lambda v,\ -\frac{1}{uvw^2}=\frac{2}{3}\lambda w.$$

Since u,v,w are nonzero we obtain that λ equals $\lambda=-\frac{1}{2u^3vw}=-\frac{1}{uv^3w}=-\frac{3}{2uvw^3}$. Then, from $\frac{1}{2u^3vw}=\frac{1}{uv^3w}$ we obtain $v^2=2u^2$; and from $\frac{1}{2u^3vw}=\frac{3}{2uvw^3}$ we obtain $w^2=3u^2$.

Using the constraint we see that $3u^2=1$, therefore $u=\frac{1}{\sqrt{3}}$, and then $v=\frac{\sqrt{2}}{\sqrt{3}}$ and $w=1$. The point is

$$\boxed{\left(\frac{1}{\sqrt{3}},\frac{\sqrt{2}}{\sqrt{3}},1\right)},$$

and the minimum volume is $\frac{3}{\sqrt{2}}=\boxed{\frac{3\sqrt{2}}{2}}$.

5. (20 points) Let E be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 12 - 2x^2 - 2y^2$ and let S be the boundary of E with outward pointing normal. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x^3 + y^2, 2yz + e^z, y^2 - z^2 \rangle$. Simplify your answer.

Solution:

Since S is a closed surface oriented outward we can use the divergence theorem. Now $\nabla \cdot \mathbf{F} = 3x^2 + 2z - 2z = 3x^2$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3x^2 dV.$$

To calculate the triple integral we use cylindrical coordinates. The paraboloids are $z = r^2$ and $z = 12 - 2r^2$. The intersection gives $r^2 = 12 - 2r^2$ so $r = 2$. Then

$$\begin{aligned}\iiint_E 3x^2 dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^{12-2r^2} 3r^2 \cos^2\theta \, r dz dr d\theta = 3 \int_0^{2\pi} \cos^2\theta d\theta \int_0^2 r^3(12 - 3r^2) dr \\ &= \pi(3r^4 - \frac{3}{6}r^6)\Big|_0^2) \\ &= \boxed{48\pi}.\end{aligned}$$

6. Let C be the curve consisting of: a line segment from $(0,0,0)$ to $(1,0,1)$ followed by the arc of a circle $x = \cos t$, $y = \sin t$, $z = 1$, $0 \leqslant t \leqslant \frac{\pi}{2}$, followed by the line segment from $(0,1,1)$ to $(0,0,0)$.

(a) (5 points) Parametrize the two line segments (with the stated orientations) and verify that C lies in the cone of equation $z = \sqrt{x^2 + y^2}$.

(b) (15 points) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -3yz\mathbf{i} + y^{10}e^{y^2}\mathbf{j} - xy\mathbf{k}$.

Solution:

(a) For the first line segment from $(0,0,0)$ to $(1,0,1)$: $\mathbf{r}(t) = t\langle 1,0,1\rangle = \langle t,0,t\rangle$, $0 \leqslant t \leqslant 1$. For the second segment from $(0,1,1)$ to $(0,0,0)$: $\mathbf{r}(s) = (1-s)\langle 0,1,1\rangle = \langle 0, 1-s, 1-s\rangle$, $0 \leqslant s \leqslant 1$.

To check that the curve lies in the cone, we verify that the parametrizations satisfy the equation of the cone. For the first line segment

$$\sqrt{x^2+y^2} = \sqrt{t^2+0^2} = t = z, \text{ so it satisfies the equation.}$$

For the second line segment

$$\sqrt{x^2+y^2} = \sqrt{0^2+(1-s)^2} = 1-s = z, \text{ so it satisfies the equation.}$$

For the arc of the circle

$$\sqrt{x^2+y^2} = \sqrt{\cos^2 t + \sin^2 t} = 1 = z, \text{ so it satisfies the equation too.}$$

(b) We use Stokes' Theorem where S is the part of the cone enclosed by the curve C. The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3yz & y^{10}e^{y^2} & -xy \end{vmatrix} = -x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}.$$

The cone $z = \sqrt{x^2+y^2}$ has equation in cylindrical coordinates $z = r$ and the surface S can be parametrized in cylindrical coordinates (or cartesian coordinates) as $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$, where $0 \leqslant \theta \leqslant \frac{\pi}{2}$ and $0 \leqslant r \leqslant 1$. Then

$$\mathbf{r}_r = \langle \cos\theta, \sin\theta, 1\rangle, \; \mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

and the cross product is $\mathbf{r}_r \times \mathbf{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle$ which is the upward pointing normal as required by the right hand rule. Then

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^1 \langle -r\cos\theta, -2r\sin\theta, 3r\rangle \cdot \langle -r\cos\theta, -r\sin\theta, r\rangle dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^2\cos^2\theta + 2r^2\sin^2\theta + 3r^2 dr d\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}} 4 + \sin^2\theta d\theta \\
&= \frac{1}{3}(2\pi + \frac{\pi}{4}) \\
&= \boxed{\frac{3\pi}{4}}.
\end{aligned}$$

7. (20 points) Let g be a function of one variable such that the derivatives g', g'' and g''' are continuous on $\mathbb{R}$. Define $f(x,y) = g''(\sqrt{x^2+y^2})$, that is, $f(x,y)$ equals the **second derivative** of g evaluated at $\sqrt{x^2+y^2}$. For the disc $D = \{(x,y)|x^2+y^2 \leqslant 9\}$ calculate

$$\iint_D xf_x + yf_y \, dA,$$

in terms of the values of g, g' and g'' at 0 and 3.

Solution:

The partial derivatives of f are

$$f_x = g'''(\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}}, \quad f_y = g'''(\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}},$$

so then

$$xf_x + yf_y = g'''(\sqrt{x^2+y^2})\frac{x^2}{\sqrt{x^2+y^2}} + g'''(\sqrt{x^2+y^2})\frac{y^2}{\sqrt{x^2+y^2}} = g'''(\sqrt{x^2+y^2})\sqrt{x^2+y^2}.$$

Writing the integral in polar coordinates we get

$$\iint_D xf_x + yf_y \, dA = \int_0^{2\pi}\int_0^3 g'''(r)r \cdot r dr d\theta = 2\pi \int_0^3 g'''(r)r^2 dr.$$

We integrate by parts with $u = r^2$, $du = 2rdr$, $dv = g'''(r)dr$, $v = g''(r)$ to get

$$\iint_D xf_x + yf_y \, dA = 2\pi\Big(g''(r)r^2\Big|_0^3 - 2\int_0^3 g''(r)rdr\Big)$$

and a new integration by parts with $u = r$, $du = dr$, $dv = g''(r)dr$, $v = g'(r)$ gives

$$\iint_D xf_x + yf_y \, dA = 2\pi\Big(g''(r)r^2\Big|_0^3 - 2\Big(g'(r)r\Big|_0^3 - \int_0^3 g'(r)dr\Big)\Big).$$

Evaluating

$$\boxed{\iint_D xf_x + yf_y \, dA = 2\pi(9g''(3) - 6g'(3) + 2g(3) - 2g(0))},$$

where we used the fundamental theorem of calculus to evaluate the integral of g'.