Solutions to the Final Exam, Math 53, Summer 2012

1. **(a)** (10 points) Let C be the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 1$ with counterclockwise orientation. Calculate $\int_C (y^2 + e^{\sqrt{x}})dx + xdy$.

   (b) (10 points) If the directional derivatives at the point $(1, 1)$ are given

$$D_{\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle} f(1, 1) = \sqrt{2}, \quad D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(1, 1) = \sqrt{3},$$

   find $f_x(1, 1)$ and $f_y(1, 1)$.

2. Let S be the surface parametrized by $\mathbf{r}(u, v) = \langle \sin u \cos u, \sin^2 u, v \rangle$ where the domain of the parameters is $D = \{(u, v) | 0 \leqslant u \leqslant \frac{\pi}{2},\ 0 \leqslant v \leqslant \sin^2 u\}$.

(a) (10 points) Find the tangent plane at the point $(x, y, z) = (\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{1}{2})$.

(b) (10 points) Calculate $\displaystyle\iint_S (x + 1) dS$.

3. (20 points) Define $\mathbf{G} = \langle 2zxe^{x^2-y^2}, -2zye^{x^2-y^2}, e^{x^2-y^2} + 2z \rangle$, $\mathbf{H} = \langle 0, x, -y \rangle$ and $\mathbf{F} = \mathbf{G} + \mathbf{H}$. Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the line segment from $(1, 2, 4)$ to $(-1, 1, 1)$.

   Hint: Calculate the line integrals for $\mathbf{G}$ and $\mathbf{H}$ separately. Use a different method for each integral.

4. (20 points) Let S be the ellipsoid of equation $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$ and let (u, v, w) be a point in S with $u > 0$, $v > 0$ and $w > 0$.

The tangent plane to S at (u, v, w) has equation $ux + \frac{vy}{2} + \frac{wz}{3} = 1$ and together with the three coordinate planes encloses a (pyramid-like) solid E whose volume equals $\frac{1}{uvw}$.

Find the point (u, v, w) as in the first paragraph such that E has the minimum possible volume. Write what that volume is.

5. (20 points) Let E be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 12 - 2x^2 - 2y^2$ and let S be the boundary of E with outward pointing normal. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x^3 + y^2, 2yz + e^z, y^2 - z^2 \rangle$. Simplify your answer.

6. Let C be the curve consisting of: a line segment from $(0,0,0)$ to $(1,0,1)$ followed by the arc of a circle $x = \cos t$, $y = \sin t$, $z = 1$, $0 \leqslant t \leqslant \frac{\pi}{2}$, followed by the line segment from $(0,1,1)$ to $(0,0,0)$.

(a) (5 points) Parametrize the two line segments (with the stated orientations) and verify that C lies in the cone of equation $z = \sqrt{x^2 + y^2}$.

(b) (15 points) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -3yz\mathbf{i} + y^{10}e^{y^2}\mathbf{j} - xy\mathbf{k}$.

7. (20 points) Let g be a function of one variable such that the derivatives g', g'' and g''' are continuous on $\mathbb{R}$. Define $f(x,y) = g''(\sqrt{x^2+y^2})$, that is, $f(x,y)$ equals the **second derivative** of g evaluated at $\sqrt{x^2+y^2}$. For the disc $D = \{(x,y)|x^2+y^2 \leqslant 9\}$ calculate

$$\iint_D x f_x + y f_y \, dA,$$

in terms of the values of g, g' and g'' at 0 and 3.