Math H54 Final Exam
December 14, 2011

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Instructions: Show all of your work, and clearly indicate your answers. Use the backs of pages as scratch paper. You will need pencils/pens and erasers, nothing more. Keep all devices capable of communication turned off and out of sight. The exam has twenty pages, including this one.

Remember: It is often possible to check your answer, and there is sometimes more than one way to solve a problem.

Problem	Your score	Possible Points
1		5
2		5
3		5
4		5
5		5
6		6
7		8
8		5
9		6
10		10
Total		60

1. (5 points) Let $A \in \mathbb{M}^{m,n}$ and $B \in \mathbb{M}^{n,p}$. Show that

$$\text{rank}(AB) \le \text{minimum}\{\text{rank}(A), \text{rank}(B)\}.$$

[Recall that the rank is the dimension of the image of the associated linear transformation. Another way to say it: the rank is the dimension of the column space.]

Step 1: Show that $\text{rank}(AB) \le \text{rank}(A)$.
Let $AB\mathbf{x} \in \text{Col}(AB)$ ($\mathbf{x} \in \mathbb{R}^p$). Then $AB\mathbf{x} = A(B\mathbf{x}) \in \text{Col}(A)$. Thus

$$\text{Col}(AB) \subset \text{Col}(A),$$

so $\text{rank}(AB) \le \text{rank}(A)$.

Step 2: Show that $\text{rank}(AB) \le \text{rank}(B)$.
By the Rank-Nullity Theorem,

$$\text{rank}(B) = p - \text{nullity}(B)$$

and

$$\text{rank}(AB) = p - \text{nullity}(AB).$$

So it suffices to show that

$$\text{nullity}(B) \le \text{nullity}(AB),$$

but this is true because clearly

$$\text{Nul}(B) \subset \text{Nul}(AB).$$

[If $\mathbf{x} \in \text{Nul}(B)$, then $AB\mathbf{x} = A(\mathbf{0}) = \mathbf{0}$.]

[It is totally fine if you write $\text{Im}(A)$ instead of $\text{Col}(A)$ and $\text{Ker}(A)$ instead of $\text{Nul}(A)$.]

Second Proof. (Ask yourself: how different is it from the first proof?)
Write

$$B = \begin{pmatrix} | & & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_p \\ | & & | \end{pmatrix},$$

where $\mathbf{b}_k \in \mathbb{R}^n$. Then by the definition of matrix multiplication we have

$$AB = \begin{pmatrix} | & & | \\ A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \\ | & & | \end{pmatrix},$$

and we have

$$A\mathbf{b}_k \in \text{Col}(A).$$

This shows that

$$\mathrm{Col(AB)} \subset \mathrm{Col}(A).$$

Similarly, but working with rows [exercise: fill in the details], we have

$$\mathrm{Row}(AB) \subset \mathrm{Row}(B).$$

Since row rank equals column rank, we thus have

$$\mathrm{rank}(AB) = \dim \mathrm{Col}(AB) = \dim \mathrm{Row}(AB) \leq \mathrm{minimum}\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

[Note: you can avoid discussing rows by taking transposes.]

Third Proof. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $\mathrm{Col}(B)$. Then, since

$$AB = \begin{pmatrix} | & & | \\ A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \\ | & & | \end{pmatrix},$$

we see that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ spans $\mathrm{Col}(AB)$. By the Toss-Out Theorem, we thus have

$$\mathrm{rank}(AB) \leq \mathrm{rank}(B) = k.$$

Similarly, but working with rows [exercise: fill in the details], we have

$$\mathrm{rank}(AB) \leq \mathrm{rank}(A).$$

Alternatively,...

Fourth Proof. Let

$$\{AB\mathbf{x}_1, \ldots, AB\mathbf{x}_N\}$$

be a basis for $\mathrm{Col}(AB)$. One can show that

$$\{B\mathbf{x}_1, \ldots, B\mathbf{x}_N\}$$

must be linearly independent [exercise]. We then use the Toss-In Theorem to get a basis for $\mathrm{Col}(B)$. Thus

$$N = \mathrm{rank}(AB) \leq \mathrm{rank}(B).$$

2. Let $(V, \langle \cdot, \cdot\cdot \rangle)$ be a two-dimensional real inner product space, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for V.

a. (2 points) Show that there exist $a, b, c \in \mathbb{R}$ such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = ax_1y_1 + b(x_1y_2 + x_2y_1) + cx_2y_2$$

for any $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2$ in V.

b. (3 points) Show that $b^2 < ac$.

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a. By the bilinearity of the inner product, for any

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 \in V$$

we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1\langle \mathbf{v}_1, \mathbf{v}_1 \rangle + x_1y_2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + x_2y_1\langle \mathbf{v}_2, \mathbf{v}_1 \rangle + x_2y_2\langle \mathbf{v}_2, \mathbf{v}_2 \rangle.$$

Let

$$\begin{aligned} a &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle, \\ b &= \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \quad (= \langle \mathbf{v}_2, \mathbf{v}_1 \rangle \text{ by the symmetry of the inner product}), \quad \text{and} \\ c &= \langle \mathbf{v}_2, \mathbf{v}_2 \rangle. \end{aligned}$$

Then indeed

$$\langle \mathbf{x}, \mathbf{y} \rangle = ax_1y_1 + b(x_1y_2 + x_2y_1) + cx_2y_2$$

for any $\mathbf{x}, \mathbf{y} \in V$.

b. By definition, an inner product is *positive definite*:

$$\langle \mathbf{x}, \mathbf{x} \rangle > 0 \qquad \text{for any} \quad \mathbf{x} \neq \mathbf{0} \text{ in } V.$$

Taking $0 \neq x_1 \in \mathbb{R}$ and $\mathbf{x} = x_1\mathbf{v}_1$, we see that

$$0 < \langle \mathbf{x}, \mathbf{x} \rangle = ax_1^2,$$

so $a > 0$.

For any $\mathbf{0} \neq \mathbf{x} \in V$ we have

$$\begin{aligned} 0 < \langle \mathbf{x}, \mathbf{x} \rangle &= ax_1^2 + 2bx_1x_2 + cx_2^2 \\ &= a\left(\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(\frac{c}{a} - \frac{b^2}{a^2}\right)x_2^2\right). \end{aligned}$$

Let $0 \neq x_2 \in \mathbb{R}$ and take $x_1 = -\frac{b}{a}x_2$. Then

$$0 < \left(\frac{c}{a} - \frac{b^2}{a^2}\right)x_2^2,$$

so $ac - b^2 > 0$.

Second Proof of (b). The Cauchy-Schwarz inequality says

$$|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle|^2 < ||\mathbf{v}_1||^2 ||\mathbf{v}_2||^2.$$

(There is strict inequality because the vectors are not multiples of each other.) That is,

$$b^2 < ac.$$

[There are variations of this proof, depending on to what extent you *prove* the Cauchy-Schwarz inequality.]

Third Proof of (b). Let $0 \neq x_2 \in \mathbb{R}$ and let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Then

$$0 < ||\mathbf{x}||^2 = ax_1^2 + 2bx_1x_2 + cx_2^2 \tag{1}$$

for all $x_1 \in \mathbb{R}$. The roots of this polynomial are

$$x_1 = \frac{-2bx_2 \pm \sqrt{4b^2x_2^2 - 4acx_2^2}}{2a}.$$

Because of (1), the polynomial can not have any *real* roots, so we must have

$$b^2 < ac.$$

Fourth Proof of (b). Let $0 \neq x_1 \in \mathbb{R}$ and take

$$\mathbf{x} = x_1\mathbf{v}_1 + \left(-\operatorname{sgn}(b)\sqrt{\frac{a}{c}}x_1\right)\mathbf{v}_2.$$

Then

$$\begin{aligned} 0 &< ax_1^2 + 2bx_1\left(-\operatorname{sgn}(b)\sqrt{\frac{a}{c}}x_1\right) + c\left(\frac{a}{c}x_1^2\right) \\ &= 2ax_1^2 - 2|b|\sqrt{\frac{a}{c}}x_1^2. \end{aligned}$$

Thus

$$|b|\sqrt{\frac{a}{c}} < a,$$

which is equivalent to

$$b^2 < ac.$$

Why does this work?

Note for the Interested:

In $\mathcal{B}$-coordinates, the inner product is of the form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Bilinearity corresponds to the fact that $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a matrix.

Symmetry corresponds to the fact that $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a *symmetric* matrix.

Positive-definiteness corresponds to the fact that

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0.$$

3. (5 points) Let T be an invertible linear transformation on a finite dimensional vector space V. Prove that if T is diagonalizable then T^{-1} is diagonalizable.

Since T is diagonalizable, it has an eigenbasis

$$\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

Say $T\mathbf{v}_j = \lambda_j \mathbf{v}_j$, $j = 1, \ldots, n$.
Since T is invertible, we have $\lambda_j \neq 0$ for all j.
Hence $T^{-1}\mathbf{v}_j = \frac{1}{\lambda_j}\mathbf{v}_j$.
Thus $\mathcal{B}$ is an eigenbasis for T^{-1}, so T^{-1} is diagonalizable.

[This was also a question on Midterm 2.]

Second (Less Elegant) Proof: Let $\mathcal{B}$ be a basis for V and let

$$A = [T]_{\mathcal{B}},$$

the matrix of T with respect to the basis $\mathcal{B}$. Since T is diagonalizable, we have that A is diagonalizable [why? this is where the proof is less elegant]. So there exists an invertible matrix P and a diagonal matrix D such that

$$A = P^{-1}DP.$$

Etc... [It's hard for me to write this, being so much worse than the first proof.]

4. (5 points) Let

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathbb{M}^{2,2}(\mathbb{R}),$$

and suppose $(a, b) \neq (0, 0)$. Apply the Gram-Schmidt process to the columns of A to obtain an orthogonal basis for the column space. Simplify your final expression as much as possible.

Let $\mathbf{w}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix} c \\ d \end{pmatrix}$.

The Gram-Schmidt process:

Let

$$\mathbf{v}_1 = \mathbf{w}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\mathbf{w}_2 \cdot \mathbf{w}_1}{||\mathbf{w}_1||^2}\mathbf{w}_1 \\
&= \begin{pmatrix} c \\ d \end{pmatrix} - \frac{(ac + bd)}{a^2 + b^2}\begin{pmatrix} a \\ b \end{pmatrix} \\
&= \frac{1}{a^2 + b^2}\begin{pmatrix} a^2c + b^2c - a^2c - abd \\ a^2d + b^2d - abc - b^2d \end{pmatrix} \\
&= \frac{ad - bc}{a^2 + b^2}\begin{pmatrix} -b \\ a \end{pmatrix} \\
&= \frac{\det(A)}{a^2 + b^2}\begin{pmatrix} -b \\ a \end{pmatrix}.
\end{aligned}$$

If $\det(A) = 0$, the columns of A are linearly dependent, and $\{\mathbf{v}_1\}$ is a basis for $\mathrm{Col}(A)$.

If $\det(A) \neq 0$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $\mathrm{Col}(A)$.

5. (5 points) Calculate the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix}.$$

[It might be easiest to use abstract properties of the determinant function.]

To simplify calculations, we follow the hint and use
(i) the fact that the determinant function is linear separately in each of the rows, and
(ii) the fact that if one row is equal to another row, the determinant is zero
to get

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix} &= \det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -4 & 0 & -4 \\ 0 & -1 & 1 & 6 \\ 0 & 4 & -1 & -3 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -4 & -28 \\ 0 & -1 & 1 & 6 \\ 0 & 0 & 3 & 21 \end{pmatrix} \\
&= (-12) \det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & -1 & 1 & 6 \\ 0 & 0 & 1 & 7 \end{pmatrix} \\
&= 0.
\end{aligned}$$

You could also do a similar computation with the *columns*.

Second Proof. I took this matrix from a linear algebra book because I think it looks nice. Using the symmetry properties of the matrix, Darsh came up with the following clever proof. It uses the fact that the determinant function is "alternating."

$$\begin{aligned}
\det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix} &= -\det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ 0 & 4 & -1 & -3 \\ -1 & -3 & 0 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & 4 & 0 & -3 \\ 0 & -3 & -1 & 4 \end{pmatrix} \\
&= -\det \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix}.
\end{aligned}$$

Since the number is equal to its negative, it must be zero.

6a. (3 points) Find a constant coefficient differential equation having the following functions as solutions:

$$t^2, \quad \sin t, \quad t^2 \sin t.$$

b. (3 points) Check your answer. [It is easiest to leave your operator factorized.]

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a. We recall that solutions of the form $t^k \sin t$, $k \in \mathbb{N}$, occur when the auxiliary polynomial has a repeated root.

We have that $\sin t$ satisfies

$$\left(\frac{d^2}{dt^2} + 1\right) \sin t = 0$$

and t^2 satisfies

$$\frac{d^3}{dt^3}(t^2) = 0,$$

so we take the constant coefficient ODE

$$\frac{d^3}{dt^3} \left(\frac{d^2}{dt^2} + 1\right)^3 x(t) = 0.$$

We could expand it out, but it's actually more convenient to leave it factorized like this.

b. Let

$$L = \frac{d^3}{dt^3} \left(\frac{d^2}{dt^2} + 1\right)^3.$$

Clearly $L(t^2) = 0$ and $L(\sin t) = 0$.

We compute

$$\frac{d}{dt}(t^2 \sin t) = 2t \sin t + t^2 \cos t$$

and

$$\frac{d^2}{dt^2}(t^2 \sin t) = 2 \sin t + 4t \cos t - t^2 \sin t,$$

so

$$\left(\frac{d^2}{dt^2} + 1\right) (t^2 \sin t) = 2 \sin t + 4t \cos t.$$

This

$$\left(\frac{d^2}{dt^2} + 1\right)^2 (t^2 \sin t) = 4 \left(\frac{d^2}{dt^2} + 1\right) (t \cos t).$$

But

$$\frac{d}{dt}(t \cos t) = \cos t - t \sin t$$

and

$$\frac{d^2}{dt^2}(t \cos t) = -2 \sin t - t \cos t,$$

so

$$\left(\frac{d^2}{dt^2}+1\right)(t\cos t) = -2\sin t.$$

Thus

$$\left(\frac{d^2}{dt^2}+1\right)^3 (t^2\sin t) = 0,$$

and so

$$L(t^2\sin t) = 0.$$

7. (8 points) Use the method of variation of parameters to find the general solution of

$$y''(x) - 3y'(x) + 2y(x) = \sin(e^{-x}).$$

[It is recommended to first write the ODE as a system of first-order ODE.]

Following the hint, we rewrite the second-order ODE as a system of first-order ODE:

$$\begin{aligned}\frac{d}{dx}\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} &= \begin{pmatrix} y'(x) \\ 3y'(x) - 2y(x) + \sin(e^{-x}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(e^{-x}) \end{pmatrix}.\end{aligned}$$

That is,

$$\mathbf{y}'(x) = A\mathbf{y}(x) + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \qquad \text{and} \qquad f(x) = \sin(e^{-x}).$$

Since the auxiliary equation is $r^2 - 3r + 2 = (r-2)(r-1) = 0$, we have a fundamental matrix

$$\mathrm{X}(x) = \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix}.$$

The inverse will be important:

$$\mathrm{X}(x)^{-1} = \begin{pmatrix} 2e^{-x} & -e^{-x} \\ -e^{-2x} & e^{-2x} \end{pmatrix}.$$

We look for a particular solution of the form

$$\mathbf{y}_p(x) = \mathrm{X}(x)\mathbf{u}(x).$$

(This is "variation of parameters.") Then

$$\begin{aligned}\mathbf{y}_p'(x) &= A\mathrm{X}(x)\mathbf{u}(x) + \mathrm{X}(x)\mathbf{u}'(x) \\ &= A\mathbf{y}_p(x) + \mathrm{X}(x)\mathbf{u}'(x),\end{aligned}$$

so, to solve the equation, we want $\mathbf{u}$ to solve

$$\begin{aligned}\mathbf{u}'(x) &= \mathrm{X}(x)^{-1}\begin{pmatrix} 0 \\ f(x) \end{pmatrix} \\ &= \begin{pmatrix} -e^{-x}\sin(e^{-x}) \\ e^{-2x}\sin(e^{-x}) \end{pmatrix}.\end{aligned}$$

Write

$$\mathbf{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}.$$

By integrating, we find that

$$u_1(x) = -\cos(e^{-x})$$

and

$$u_2(x) = e^{-x}\cos(e^{-x}) - \sin(e^{-x}).$$

Thus the general solution is

$$\begin{aligned} y(x) &= c_1 e^x + c_2 e^{2x} - \cos(e^{-x})e^x + (e^{-x}\cos(e^{-x}) - \sin(e^{-x}))e^{2x} \\ &= c_1 e^x + c_2 e^{2x} - e^{2x}\sin(e^{-x}). \end{aligned}$$

8. (5 points) In class we proved the following theorem:

Theorem 1 *The initial boundary value problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < L, \end{cases}$$

has at most one twice-continuously-differentiable solution.

Give the main steps of the proof. [You do not need to give all the details.]

Let u and v be two solutions and let $w = u - v$. We are to show that $w \equiv 0$.

The main idea is to define the "energy" of the wave at time t to be

$$E(t) = \frac{1}{2} \int_0^L \left[\left(\frac{\partial w}{\partial t} \right)^2 + \alpha^2 \left(\frac{\partial w}{\partial x} \right)^2 \right] dx.$$

After differentiating under the integral sign, integrating by parts, and using that w solves the wave equation, we find that $E'(t) = 0$ for all t. Thus $E(t) = E$ is a constant. Plugging in $t = 0$ we see that $E = 0$.

From the definition of $E(t)$, we then see that

$$\frac{\partial w}{\partial t} \equiv \frac{\partial w}{\partial x} \equiv 0,$$

so w is a constant. Plugging in $t = 0$ we see that $w \equiv 0$.

9. (6 points) Let $L > 0$. Find the Fourier sine series of the function

$$H(x) = \frac{L - x}{L}$$

on the interval $[0, L]$.

The Fourier sine series of H on $[0, L]$ is

$$H(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L H(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 1, 2, 3, \ldots.$$

That is,

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= 2\int_0^1 (1 - y)\sin(n\pi y)\,dy \\ &= (I) + (II). \end{aligned}$$

For the first term:

$$\begin{aligned} (I) &= 2\int_0^1 \sin(n\pi y)\,dy \\ &= -\frac{2}{n\pi}\int_0^1 \frac{d}{dy}\cos(n\pi y)\,dy \\ &= -\frac{2}{n\pi}[\cos(n\pi) - 1] \\ &= \frac{2}{n\pi} - \frac{2}{n\pi}(-1)^n. \end{aligned}$$

For (II) we need to integrate by parts:

$$\begin{aligned} (II) &= -2\int_0^1 y\sin(n\pi y)\,dy \\ &= -2\left[-\frac{1}{n\pi}\cos(n\pi) + \int_0^1 \frac{1}{n\pi}\cos(n\pi y)\,dy\right] \\ &= \frac{2}{n\pi}(-1)^n - \frac{2}{n\pi}\int_0^1 \frac{1}{n\pi}\frac{d}{dy}(\sin(n\pi y))\,dy \\ &= \frac{2}{n\pi}(-1)^n - \frac{2}{(n\pi)^2}[0] \\ &= \frac{2}{n\pi}(-1)^n. \end{aligned}$$

Putting the two parts together, we get

$$b_n = \frac{2}{n\pi}.$$

Second Proof. Splitting the integral into two parts is unnecessary. I do, however, get confused by sign changes, so let me be clear and write out *why* integration by parts works:

$$\begin{aligned}
b_n &= 2\int_0^1 (1-y)\sin(n\pi y)\,dy \\
&= \int_0^1 \left[\frac{d}{dy}\left((1-y)\left(\frac{-2}{n\pi}\right)\cos(n\pi y)\right) + \left(\frac{-2}{n\pi}\right)\cos(n\pi y)\right] dy \\
&= \left[(0) - \left(\frac{-2}{n\pi}\right)\right] - \frac{2}{n\pi}\int_0^1 \cos(n\pi y)\,dy \\
&= \frac{2}{n\pi}.
\end{aligned}$$

10. (10 points) Let $L > 0$, $\alpha > 0$, and assume $L \neq n\pi\alpha$ for any $n \in \mathbb{N}$. Solve the problem of "waving a rope tied to a doorknob":

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, & 0 < x < L,\ t > 0, \\ u(0,t) = \cos t, \quad u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < L. \end{cases} \tag{2}$$

[See the final page for hints.]

[Actually, for the solutions, I will post the hints here:]

Hints for Problem 10:

Try to find u of the form

$$u(x,t) = v(x,t) + w(x,t),$$

where v solves a system of the form

$$\begin{cases} \frac{\partial^2 v}{\partial t^2}(x,t) = \alpha^2 \frac{\partial^2 v}{\partial x^2}(x,t) + h(x,t), & 0 < x < L,\ t > 0, \\ v(0,t) = 0, \quad v(L,t) = 0, & t > 0, \\ v(x,0) = F(x), \quad \frac{\partial v}{\partial t}(x,0) = G(x), & 0 < x < L, \end{cases}$$

for some functions h, F, and G. Then, to solve this system, look for v of the form

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

[And possibly use Problem 9.]

Following the hint, we want

$$\cos t = u(0,t) = v(0,t) + w(0,t) = w(0,t)$$

and

$$0 = u(L,t) = v(L,t) + w(L,t) = w(L,t),$$

which suggests that we take

$$w(x,t) = \left(\frac{L-x}{L}\right) \cos t.$$

[I was suggesting this by asking Problem 9.]

Then we want v to solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 w}{\partial t^2} \\ &= \alpha^2 \frac{\partial^2 v}{\partial x^2} + \alpha^2 \frac{\partial^2 w}{\partial x^2} \\ &= \alpha^2 \frac{\partial^2 u}{\partial x^2}. \end{aligned}$$

So we want v to solve

$$\frac{\partial^2 v}{\partial t^2} = \alpha^2 \frac{\partial^2 v}{\partial x^2} + \left(\frac{L-x}{L}\right)\cos t.$$

That is, including all the conditions, we want v to solve

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = \alpha^2 \frac{\partial^2 v}{\partial x^2} + \left(\frac{L-x}{L}\right)\cos t \\ v(0,t) = v(L,t) = 0 \\ v(x,0) = f(x) - \left(\frac{L-x}{L}\right) =: F(x) \\ \frac{\partial v}{\partial t}(x,0) = g(x) =: G(x). \end{cases} \tag{3}$$

Again following the hint, we look for v of the form

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

Then we want

$$\sum_{n=1}^{\infty} v_n''(t) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty}\left[-\left(\frac{\alpha n\pi}{L}\right)^2 v_n(t)\right]\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}[b_n \cos t]\sin\left(\frac{n\pi x}{L}\right),$$

where the b_n are the Fourier sine coefficients for the function $\frac{L-x}{L}$ on the interval $[0, L]$.

This suggests that we try solving the ODE

$$v_n''(t) = -\left(\frac{\alpha n\pi}{L}\right)^2 v_n(t) + b_n \cos t.$$

By the method of undetermined coefficients, we look for a particular solution of the form

$$\beta_n \cos t.$$

Since we are assuming $L \neq \alpha n\pi$ (which simplifies things), we find that

$$\beta_n = \frac{b_n}{\left(\frac{\alpha n\pi}{L}\right)^2 - 1}.$$

Thus the general solution is

$$v_n(t) = c_{1n}\cos\left(\frac{\alpha n\pi}{L}t\right) + c_{2n}\sin\left(\frac{\alpha n\pi}{L}t\right) + \left[\frac{b_n}{\left(\frac{\alpha n\pi}{L}\right)^2 - 1}\right]\cos t.$$

So

$$v(x,t) = \sum_{n=1}^{\infty}\left[c_{1n}\cos\left(\frac{\alpha n\pi}{L}t\right) + c_{2n}\sin\left(\frac{\alpha n\pi}{L}t\right) + \left[\frac{b_n}{\left(\frac{\alpha n\pi}{L}\right)^2 - 1}\right]\cos t\right]\sin\left(\frac{n\pi x}{L}\right).$$

For the initial values, we want

$$F(x) = v(x,0) = \sum_{n=1}^{\infty} \left[c_{1n} + \frac{b_n}{\left(\frac{\alpha n \pi}{L}\right)^2 - 1} \right] \sin\left(\frac{n\pi x}{L}\right),$$

so the Fourier sine coefficients of F determine the c_{1n}.

And we also want

$$G(x) = \frac{\partial v}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_{2n} \left(\frac{\alpha n \pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right),$$

so the Fourier sine coefficients of G determine the c_{2n}.

Hence we have found the solution $v(x,t)$ of (3), and so

$$u(x,t) = v(x,t) + \left(\frac{L-x}{L}\right) \cos t$$

is the solution of (2).

Note for the Interested:

What if we allow $L = \alpha n\pi$ for some $n \in \mathbb{N}$?
Then, for that one value of n, we are to solve

$$v_n''(t) + v_n(t) = b_n \cos t.$$

We look for a solution of the form

$$v_p(t) = \beta t \sin t.$$

(This was my second guess. My first had cosine instead of sine, but that didn't work.)

Then

$$v_p''(t) + v_p(t) = 2\beta \cos t,$$

so we take

$$\beta = b_n/2.$$

So for this one value of n, the general solution is

$$v_n(t) = c_{1n} \cos t + c_{2n} \sin t + \frac{b_n}{2} t \sin t.$$

In determining the c_{1n} and c_{2n} to satisfy the initial conditions, everything looks the same except now we take c_{1n} (for this one value of n) to be the nth Fourier sine coefficient of F.

I hope you had fun with this problem—I thought of it while staring at the forty-foot-long rope in my office.

If anyone manages to make a computer animation of the solutions, let me know! (Of course, you would have to make choices of f and g, for example, both identically zero.)