Math H54 Midterm 2
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**Instructions:** Show all of your work, and clearly indicate your answers. Use the backs of pages as scratch paper. You will need pencils/pens and erasers, nothing more. Keep all devices capable of communication turned off and out of sight. The exam has eleven pages, including this one. (Some pages are blank.)

**Remember:** It is often possible to check your answer.

| Problem | Your score | Possible Points |
|---|---|---|
| 1 | | 4 |
| 2 | | 5 |
| 3 | | 4 |
| 4 | | 6 |
| 5 | | 5 |
| 6 | | 6 |
| Total | | 30 |

1. (4 points) A linear transformation from a vector space $V$ to $\mathbb{R}$ is called a *linear functional* on $V$. Let $f$ be a linear functional on $\mathbb{R}^n$. Show that there exists a unique vector $\mathbf{v} \in \mathbb{R}^n$ such that

$$f(\mathbf{u}) = \mathbf{u} \cdot \mathbf{v} \qquad \text{for all } \mathbf{u} \in \mathbb{R}^n.$$

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**Existence:** Define $\mathbf{v} \in \mathbb{R}^n$ to be the vector with $j$th entry

$$v_j := f(\mathbf{e}_j), \qquad j = 1, \ldots, n,$$

where $\mathbf{e}_j$ is the $j$th standard basis vector. Then for all $\mathbf{u} \in \mathbb{R}^n$ we have

$$\begin{aligned} f(\mathbf{u}) &= f\left(\sum_{j=1}^{n} u_j \mathbf{e}_j\right) \\ &= \sum_{j=1}^{n} u_j f(\mathbf{e}_j) \\ &= \sum_{j=1}^{n} u_j v_j \\ &= \mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

**Uniqueness:** Say $f(\mathbf{u}) = \mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ for all $\mathbf{u} \in \mathbb{R}^n$. Then $f(\mathbf{e}_j) = v_j = w_j$ for all $j$, so $\mathbf{v} = \mathbf{w}$.

2. (5 points) Is the matrix $A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ diagonalizable? Prove your answer. If it is diagonalizable, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = P^{-1}DP$.

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Since $A$ is upper-triangular, the eigenvalues are the diagonal entries $\lambda = 1, 3$. We now find eigenvectors:

$\lambda = 1$:

$$\ker(A - I) = \ker \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$\lambda = 3$:

$$\ker(A - 3I) = \ker \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}.$$

We only have two linearly independent eigenvectors, which is not enough for an eigenbasis. Thus $A$ is not diagonalizable.

3. (4 points) Let $T$ be an invertible linear transformation on a finite dimensional vector space $V$. Prove that if $T$ is diagonalizable then $T^{-1}$ is diagonalizable.

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Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis for $T$.

Say $T\mathbf{v}_j = \lambda_j \mathbf{v}_j$, $j = 1, \ldots, n$.

Since $T$ is invertible, we have $\lambda_j \neq 0$ for all $j$.

Hence $T^{-1}\mathbf{v}_j = \frac{1}{\lambda_j}\mathbf{v}_j$.

Thus $\mathcal{B}$ is an eigenbasis for $T^{-1}$, so $T^{-1}$ is diagonalizable.

4a. (4 points) Find the least squares solution(s) of the system of linear equations

$$\begin{cases} x + 2y + 3z = 6 \\ x + 2y + 3z = 12 \\ x + y + z = 1. \end{cases}$$

b. (2 points) Check that your solutions satisfy the normal equations.

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(a) We let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 12 \\ 1 \end{pmatrix}.$$

To find an especially nice basis for $\text{Col}(A)$, we column reduce:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

so

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\text{Col}(A)$. (One can check this: express each column of $A$ as a linear combination of the $\mathcal{B}$-vectors.) Note that $\mathcal{B}$ is already an orthogonal basis.

Thus

$$\begin{aligned} \hat{\mathbf{b}} &= \frac{\mathbf{b} \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\mathbf{b} \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 \\ &= \begin{pmatrix} 9 \\ 9 \\ 1 \end{pmatrix}. \end{aligned}$$

Now we solve $A\mathbf{x} = \hat{\mathbf{b}}$:

$$\begin{pmatrix} 1 & 2 & 3 & 9 \\ 1 & 2 & 3 & 9 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -7 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A particular solution of the inhomogeneous equation is

$$\hat{\mathbf{x}}_p = \begin{pmatrix} -7 \\ 8 \\ 0 \end{pmatrix}.$$

And the general solution of the homogeneous equation is

$$\hat{\mathbf{x}}_h = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

So the least square solutions are all of the form

$$\hat{\mathbf{x}} = \begin{pmatrix} -7 \\ 8 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

(b) We compute

$$A^T A = \begin{pmatrix} 3 & 5 & 7 \\ 5 & 9 & 13 \\ 7 & 13 & 19 \end{pmatrix}$$

and

$$A^T \mathbf{b} = \begin{pmatrix} 19 \\ 37 \\ 55 \end{pmatrix}.$$

And

$$\begin{aligned} A^T A \hat{\mathbf{x}} &= \begin{pmatrix} 3 & 5 & 7 \\ 5 & 9 & 13 \\ 7 & 13 & 19 \end{pmatrix} \begin{pmatrix} -7 \\ 8 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 19 \\ 37 \\ 55 \end{pmatrix}, \end{aligned}$$

which is what we expected.

5. (5 points) Suppose that the Gram-Schmidt process applied to the basis $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for $\mathbb{R}^n$ results in the orthogonal basis $\mathcal{B}' = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $U \in \mathbb{M}^{n,n}$ have orthonormal columns. Prove that the Gram-Schmidt process applied to $U\mathcal{B} := \{U\mathbf{x}_1, \ldots, U\mathbf{x}_n\}$ results in $U\mathcal{B}' := \{U\mathbf{v}_1, \ldots, U\mathbf{v}_n\}$.

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We first recall that $U^T U = I$.

The hypothesis says that $\mathbf{v}_1 = \mathbf{x}_1$ and

$$\mathbf{v}_j = \mathbf{x}_j - \sum_{k=1}^{j-1} \frac{\mathbf{x}_j \cdot \mathbf{v}_k}{||\mathbf{v}_k||^2} \mathbf{v}_k \qquad \text{for j=2,...,n.}$$

We now apply the Gram-Schmidt process to $U\mathcal{B}$, the first step being

$$\mathbf{w}_1 := U\mathbf{x}_1 = U\mathbf{v}_1.$$

We prove the result by induction. Assume that the $m$th step of the Gram-Schmidt process results in $\mathbf{w}_m = U\mathbf{v}_m$. We already saw that this holds for $m = 1$. Then

$$\begin{aligned}
\mathbf{w}_{m+1} &:= U\mathbf{x}_{m+1} - \sum_{j=1}^{m} \frac{(U\mathbf{x}_{m+1}) \cdot (U\mathbf{v}_j)}{||U\mathbf{v}_j||^2} U\mathbf{v}_j \\
&= U\mathbf{x}_{m+1} - \sum_{j=1}^{m} \frac{\mathbf{x}_{m+1} \cdot \mathbf{v}_j}{||\mathbf{v}_j||^2} U\mathbf{v}_j \\
&= U\left(\mathbf{x}_{m+1} - \sum_{j=1}^{m} \frac{\mathbf{x}_{m+1} \cdot \mathbf{v}_j}{||\mathbf{v}_j||^2} \mathbf{v}_j\right) \\
&= U\mathbf{v}_{m+1}.
\end{aligned}$$

Hence the $m = 1$ case implies the $m = 2$ case, which implies the $m = 3$ case, which implies the $m = 4$ case, which ...implies the $m = n$ case. That is, we are done by induction.

6. Let $V$ be an $n$-dimensional vector space, let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for $V$, and let $T : V \to V$ be the linear transformation such that $T\mathbf{b}_1 = \mathbf{0}$ and $T\mathbf{b}_j = \mathbf{b}_{j-1}$ for $j = 2, \ldots, n$.

a. (2 points) Find the matrix $A = [T]_{\mathcal{B}}$ of $T$ with respect to the basis $\mathcal{B}$.

b. (2 points) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

c. (2 points) Let $S$ be any linear transformation on $V$ such that $S^n = 0$ and $S^{n-1} \neq 0$. Prove that there exists an ordered basis $\mathcal{B}'$ for $V$ such that $[S]_{\mathcal{B}'} = A$, where $A$ is the matrix from part (a).

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(a) We have

$$\begin{aligned} A &= \begin{bmatrix} | & & | \\ [T\mathbf{b}_1]_{\mathcal{B}} & \cdots & [T\mathbf{b}_n]_{\mathcal{B}} \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & 1 \\ 0 & 0 & 0 & 0 & 0 \cdots & 0 \end{bmatrix} \end{aligned}$$

That is, $A$ has entries

$$A_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) We compute

$$\begin{aligned} T\left(\sum_{j=1}^n c_j \mathbf{b}_j\right) &= \sum_{j=2}^n c_j \mathbf{b}_{j-1} \\ T^2\left(\sum_{j=1}^n c_j \mathbf{b}_j\right) &= \sum_{j=3}^n c_j \mathbf{b}_{j-2} \\ &\vdots \\ T^{n-1}\left(\sum_{j=1}^n c_j \mathbf{b}_j\right) &= \sum_{j=n}^n c_j \mathbf{b}_{j-n+1} = c_n \mathbf{b}_1. \end{aligned}$$

This last vector is not zero if, say, $c_n = 1$. And finally

$$T^n\left(\sum_{j=1}^n c_j \mathbf{b}_j\right) = \mathbf{0}.$$

(c) Since $S^{n-1} \neq 0$, there exists some $\mathbf{v}_0 \in V$ such that $S^{n-1}\mathbf{v}_0 \neq \mathbf{0}$. Now let

$$\begin{aligned} \mathcal{B}' &= \{S^{n-1}\mathbf{v}_0, S^{n-2}\mathbf{v}_0, \ldots, S\mathbf{v}_0, \mathbf{v}_0\} \\ &=: \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}. \end{aligned}$$

Then $S\mathbf{w}_1 = \mathbf{0}$ and $S\mathbf{w}_j = \mathbf{w}_{j-1}$ for $j = 2, \ldots, n$.

**Claim.** $\mathcal{B}'$ is a basis.

*Proof* It suffices to show that it is a linearly independent set.

Suppose

$$\sum_{j=1}^{n} c_j S^{n-j} \mathbf{v}_0 = \mathbf{0}.$$

Then

$$\begin{aligned} \mathbf{0} &= S^{n-1} \left( \sum_{j=1}^{n} c_j S^{n-j} \mathbf{v}_0 \right) \\ &= c_n S^{n-1} \mathbf{v}_0, \end{aligned}$$

so $c_n = 0$.

Suppose by induction that $c_{k+1} = c_{k+2} = \cdots = c_n = 0$. (This is true for $k = n-1$.) Then

$$\begin{aligned} \mathbf{0} &= S^{k-1} \left( \sum_{j=1}^{k} c_j S^{n-j} \mathbf{v}_0 \right) \\ &= \sum_{j=1}^{k} c_j S^{n+k-j-1} \mathbf{v}_0 \\ &= c_k S^{n-1} \mathbf{v}_0, \end{aligned}$$

so $c_k = 0$.

Hence by induction

$$c_1 = c_2 = \cdots = c_n = 0.$$

Thus it is a basis and we can see as before that

$$\begin{aligned} [S]_{\mathcal{B}'} &= \begin{bmatrix} | & & | \\ [S\mathbf{w}_1]_{\mathcal{B}'} & \ldots & [S\mathbf{w}_n]_{\mathcal{B}'} \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & 1 \\ 0 & 0 & 0 & 0 & 0\cdots & 0 \end{bmatrix} \\ &= A. \end{aligned}$$