Math H54 Midterm 1
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Name: ______________________________

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Instructions: Show all of your work, and clearly indicate your answers. Use the backs of pages as scratch paper. You will need pencils/pens and erasers, nothing more. Keep all devices capable of communication turned off and out of sight. The exam has eight pages, including this one.

Remember: It is often possible to check your answer, and there is sometimes more than one way to solve a problem.

Strategic Guidance: The problems are arranged in order of increasing difficulty and decreasing point value. Problem 5 might only be the difference between an “A” and an “A+.”

Problem	Your score	Possible Points
1		6
2		6
3		6
4		4
5		3
Total		25

1. (6 points) Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set or find $c_1, c_2, c_3 \in \mathbb{R}$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$.

We row-reduce the matrix with column vectors $\mathbf{v}_j$:

$$\begin{aligned} \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & 2 \\ -3 & 2 & -2 \end{pmatrix} &\to \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \\ &\to \begin{pmatrix} 3 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now take, for example, $c_1 = 2$, $c_2 = 2$, and $c_3 = -1$. Then we can check:

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= 2\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

(The principle we are using is that $A\mathbf{x} = \mathbf{0}$ and $\text{rref(A)}\mathbf{x} = \mathbf{0}$ have the same solutions.)

2. (6 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find an invertible matrix P and a matrix R in reduced-row echelon form such that $PA = R$.

We row-reduce the augmented matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & r_1 \\ 1 & 2 & -1 & -1 & 0 & r_2 \\ 0 & 0 & 1 & 4 & 0 & r_3 \\ 2 & 4 & 1 & 10 & 1 & r_4 \\ 0 & 0 & 0 & 0 & 1 & r_5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & r_1 \\ 0 & 0 & -1 & -4 & 0 & r_2 - r_1 \\ 0 & 0 & 1 & 4 & 0 & r_3 \\ 2 & 4 & 1 & 10 & 0 & r_4 - r_5 \\ 0 & 0 & 0 & 0 & 1 & r_5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & r_1 \\ 0 & 0 & 0 & 0 & 0 & r_2 - r_1 + r_3 \\ 0 & 0 & 1 & 4 & 0 & r_3 \\ 0 & 0 & 1 & 4 & 0 & r_4 - r_5 - 2r_1 \\ 0 & 0 & 0 & 0 & 1 & r_5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & r_1 \\ 0 & 0 & 1 & 4 & 0 & r_3 \\ 0 & 0 & 0 & 0 & 1 & r_5 \\ 0 & 0 & 0 & 0 & 0 & r_4 - r_5 - 2r_1 - r_3 \\ 0 & 0 & 0 & 0 & 0 & r_2 - r_1 + r_3 \end{pmatrix}$$

The coefficient matrix here is the matrix R. Let

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Now we can check:

$$PA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & 1 & -1 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

Note: You can do the same calculations in different notation by row-reducing

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 4 & 1 & 10 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This way involves a bit more writing, but in principle it is not any harder...

Note that the matrix P is not unique.

3. (6 points) It is a fact that if $A, B, C, D \in \mathbb{M}^{n,n}$ are such that $AC = CA$, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

(The matrix on the left is $2n \times 2n$ and the matrix on the right is $n \times n$.) Give an example of $A, B, C, D \in \mathbb{M}^{2,2}$ such that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq \det(AD - CB).$$

Explicitly compute both determinants, showing that they are not equal.

One example is given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} &= \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= (-1)^2 \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= -1. \end{aligned}$$

On the other hand,

$$\begin{aligned} AD - CB &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

so

$$\det(AD - CB) = 0.$$

There are many other examples (so this will be a difficult problem to grade). Simply start with two matrices $A, C \in \mathbb{M}^{2,2}$ such that $AC \neq CA$, then try to cook up $B, D \in \mathbb{M}^{2,2}$ to give the desired property. Choose A, B, C, D to be as simple as possible, to make all the computations easy.

Here's another example:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$$

and

$$\det(AD - CB) = 0.$$

4. For any $A \in \mathbb{M}^{m,n}$ it is true that

(I) If A is surjective (onto), then the transpose A^T is injective (one-to-one), and

(II) If A is injective, then A^T is surjective.

a. (3 points) Prove either (I) or (II).

b. (1 point) Give a brief description of how you might try proving the other direction. (You may appeal to geometric intuition.)

You only needed to completely prove *one* of the above statements, then just give a sketch of the other statement. For completeness, I'll give two proofs of *each* of the statements.

First Proof.

In either case, there exists an invertible matrix $P \in \mathbb{M}^{m,m}$ and a matrix $R \in \mathbb{M}^{m,n}$ in reduced row echelon form such that $PA = R$.

(I): Assume A is surjective. We first prove that R is surjective. Let $\mathbf{b} \in \mathbb{R}^m$. Since A is surjective, there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = P^{-1}\mathbf{b}$. Thus

$$R\mathbf{x} = PA\mathbf{x} = PP^{-1}\mathbf{b} = \mathbf{b},$$

so R is surjective.

Since R is surjective and is in reduced row echelon form, it must have a pivot in every row (otherwise there is a row of zeros). Thus R^T has a pivot in every column, hence is injective.

Now we will see that A^T is injective. Suppose $A^T\mathbf{x} = \mathbf{0}$. Then

$$R^T(P^T)^{-1}\mathbf{x} = A^T\mathbf{x} = \mathbf{0}.$$

Since R^T is injective, $\mathbf{x} = \mathbf{0}$. Hence A^T is injective.

(II): Assume A is injective. We first prove that R is injective. Suppose $R\mathbf{x} = \mathbf{0}$. Thus, $PA\mathbf{x} = \mathbf{0}$, hence $A\mathbf{x} = \mathbf{0}$, hence $\mathbf{x} = \mathbf{0}$.

The columns of R are thus linearly independent. Since R is in reduced row echelon form, it thus has a pivot in every column. Thus R^T has a pivot in every row, so is surjective.

Now we will show that A^T is surjective. Let $\mathbf{b} \in \mathbb{R}^n$. Then there exists some $\mathbf{x} \in \mathbb{R}^m$ such that $R^T\mathbf{x} = \mathbf{b}$. Hence

$$A^T(P^T\mathbf{x}) = \mathbf{x},$$

so A^T is surjective.

Second Proof. (More beautiful, more elegant...)

(I): Let $\mathbf{e}_j \in \mathbb{R}^m$ be the vector with a 1 in the jth position and 0's everywhere else. Since A is surjective, there exists $\mathbf{v}_j \in \mathbb{R}^n$ such that $A\mathbf{v}_j = \mathbf{e}_j$. Let $B \in \mathbb{M}^{n,m}$ be the matrix whose jth column is $\mathbf{v}_j$. Then $AB = I_m$. Hence $B^TA^T = I_m$.

Now suppose $A^T\mathbf{x} = A^T\mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Then

$$\mathbf{x} = B^TA^T\mathbf{x} = B^TA^T\mathbf{y} = \mathbf{y},$$

which shows that A^T is injective.

(II): Suppose A^T is *not* surjective. Then

$$\{A^T\mathbf{x};\ \mathbf{x} \in \mathbb{R}^m\} \neq \mathbb{R}^n.$$

There exists some $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n$ that is perpendicular to the whole set $\{A^T\mathbf{x};\ \mathbf{x} \in \mathbb{R}^m\}$. That is,

$$\mathbf{y} \cdot A^T\mathbf{x} = \mathbf{0} \quad \text{for all } \mathbf{x} \in \mathbb{R}^m.$$

Hence

$$A\mathbf{y} \cdot \mathbf{x} = \mathbf{0} \quad \text{for all } \mathbf{x} \in \mathbb{R}^m.$$

Thus $A\mathbf{y} = \mathbf{0}$. Since $\mathbf{y} \neq 0$, this shows that A is not injective.

5. (3 points) In class I stated but did not prove the fact that there is a unique function from $\mathbb{M}^{n,n}$ to $\mathbb{R}$ that is multilinear, alternating, and normalized. Assume existence and prove uniqueness. That is, assume there is such a function and call it "det." Show that if $D : \mathbb{M}^{n,n} \to \mathbb{R}$ is another such function, then in fact $D = \det$. (*Hint*: The proof I have in mind is similar to our proof that $\det(AB) = \det(A)\det(B)$.)

Let $A \in \mathbb{M}^{n,n}$. Then there exist elementary matrices E_j and a matrix R in reduced-row echelon form such that

$$E_p E_{p-1} \cdots E_2 E_1 R = A.$$

Let $D : \mathbb{M}^{n,n} \to \mathbb{R}$ be multilinear, alternating, and normalized. For any $B \in \mathbb{M}^{n,n}$, we have:

(a) Since D is alternating, when E_j represents "interchange" we have

$$D(E_j B) = -D(B).$$

(b) Since D is multilinear, when E_j represents "replacement" we have

$$D(E_j B) = D(B).$$

(a) Since D is multilinear, when E_j represents "multiplication of a row by $c \neq 0$" we have

$$D(E_j B) = cD(B).$$

In the special case when $D = \det$ and $B = I$, this says

$$\det(E_j) = \begin{cases} -1 & \text{if } E_j \text{ represents interchange} \\ 1 & \text{if } E_j \text{ represents replacement} \\ c & \text{if } E_j \text{ represents multiplication of a row by } c \neq 0. \end{cases}$$

So we have

$$D(E_j B) = \det(E_j) D(B) \quad \text{for any } B \in \mathbb{M}^{n,n}.$$

Thus

$$\begin{aligned} D(A) &= D(E_p E_{p-1} \cdots E_2 E_1 R) \\ &= \det(E_p) \cdots \det(E_1) D(R) \\ &= \det(E_p \cdots E_1) D(R). \end{aligned}$$

If $R = I$, then

$$D(A) = \det(A)$$

since D is normalized.

If $R \neq I$, then it has a row of zeros. Thus

$$D(A) = 0 = \det(A)$$

since D is alternating.

So in all cases

$$D(A) = \det(A)$$

for all $A \in \mathbb{M}^{n,n}$. That is, $D = \det$.